On polynomial free-by-cyclic groups

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Abstract

A free-by-cyclic group can often be viewed as a mapping torus of a free group automorphism (monodromy) in multiple ways. What dynamical properties must these monodromies share, and to what extent are they invariant under quasi-isometries? We give a new proof using cyclic splittings that the polynomial growth of a monodromy is a geometric invariant of the free-by-cyclic group; we also characterise the degree of polynomial growth using slender splittings. For exponential growth, we conjecture that the nesting of attracting laminations is a geometric invariant.

These are notes for a minicourse I taught at CRM-Montréal in May 2023. The minicourse consisted of three lectures that covered: dynamics of free group automorphisms (Section 2); group invariance of growth type (Section 3); and geometric invariance of growth type (Sections 4 and 5). We have included some new results, proofs of folklore statements, and new proofs of published results.

1 Introduction

A group is free-by-cyclic if it has a free normal subgroup whose quotient group is cyclic; we will insist the free subgroup is finitely generated and not trivial and the quotient is infinite. In the last two sections, we will also assume the free subgroup is not cyclic to rule out, for convenience, the fundamental group of a torus or Klein bottle.

Rephrasing the definition, a free-by-cyclic group $\mathbb{G}$ sits in a short exact sequence

$$1 \to \mathbb{F} \to \mathbb{G} \to \mathbb{Z} \to 1, \text{ where } \mathbb{F} \text{ is finitely generated free and not trivial.}$$

Since $\mathbb{Z}$ is free, the short exact sequence splits, and $\mathbb{G} \cong \mathbb{F} \rtimes_\Phi \mathbb{Z}$ for some automorphism $\Phi\colon \mathbb{F} \to \mathbb{F}$ that is well-defined up to composition with an inner automorphism of $\mathbb{F}$. There are of course plenty of things one can say about $\mathbb{G}$; we focus on how the dynamics of the outer automorphism $\phi = [\Phi]$ relate to the algebraic and geometric properties of $\mathbb{G}$.

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This is particularly interesting when $\mathbb{G}$ has two finitely generated normal free subgroups $\mathbb{F} \neq \mathbb{F}'$ with cyclic quotients, and we get an isomorphism $\mathbb{F} \rtimes_\Phi \mathbb{Z} \cong \mathbb{F}' \rtimes_\Psi \mathbb{Z}$. What dynamical properties must the outer automorphisms ϕ and $\psi = [\Psi]$ have in common?

Question 1. Generally, what must ϕ, ψ have in common if $\mathbb{G} \cong \mathbb{F} \rtimes_\Phi \mathbb{Z}$ is quasi-isometric to $\mathbb{G}' \cong \mathbb{F}' \rtimes_\Psi \mathbb{Z}$? (See Section 4 for a definition of quasi-isometry.)

Our main result states that the two outer automorphisms must share growth type.

Corollary 4.9. *If $\mathbb{G}, \mathbb{G}'$ are quasi-isometric and ϕ is polynomially growing, then so is ψ.*

It is then a theorem of Nataša Macura that they must also share their degrees of growth $\deg(\psi) = \deg(\phi)$ (Theorem 4.10). *Polynomial growth* and *degree* are defined in Section 2. This corollary was recently proven using relative hyperbolicity (Theorem 5.2), while our proof uses cyclic splittings. Along the way, we give a very short proof that growth type is a group invariant (Theorem 3.2). The previously known proof of group invariance through relative hyperbolicty was a bit overkill.

We have included characterisations and observations that may be known to experts but do not explicitly appear elsewhere; here are three such observations.

Lemma 3.5. *If $\mathbb{G}' \leq \mathbb{G}$ is a free-by-cyclic subgroup, then $\mathbb{G}' \cap \mathbb{F}$ is finitely generated.*

Lemma 4.1. *Free-by-cyclic subgroups of $\mathbb{G}$ are undistorted.*

By Feighn–Handel's coherence result [FH99, Prop. 2.3], we can replace "free-by-cyclic subgroup" with "finitely generated noncyclic subgroup with Euler characteristic 0".

Lemma 4.2. *Slender subgroups of $\mathbb{G}$ are undistorted.*

A finitely generated subgroup of $\mathbb{G}$ is undistorted if the inclusion is a q.i.-embedding; for instance, $\mathbb{F}$ is distorted in $\mathbb{G}$ when ϕ is exponentially growing or polynomially growing with degree at least 2. A group is slender if every subgroup is finitely generated; for example, finitely generated abelian groups are slender. Slender subgroups of $\mathbb{F}$ are cyclic, and slender subgroups of $\mathbb{G}$ are cyclic or $\mathbb{Z} \rtimes \mathbb{Z}$. We also consider splittings of $\mathbb{G}$ over slender subgroups and the *depths* of *slender hierarchies* for $\mathbb{G}$ as defined in Section 3.

Lemma 3.1. *If $\mathbb{F}$ is not cyclic, then slender (resp. cyclic) splittings of $\mathbb{G}$ are naturally in one-to-one correspondence with ϕ-fixed cyclic (resp. free) splittings of $\mathbb{F}$.*

This correspondence allows us to give a group invariant characterisation of $\deg(\phi)$ when $\mathbb{G}$ is polynomial, i.e. ϕ is polynomially growing.

Corollary 3.3. $\deg(\phi)$ *is the minimal depth of slender hierarchies for polynomial $\mathbb{G}$.*

With the hope of turning this characterisation into a geometric invariant, we guess that slender splittings of $\mathbb{G}$ are preserved by quasi-isometries.

Conjecture 4.5. *If $\mathbb{G}, \mathbb{G}'$ are quasi-isometric, then a slender splitting of $\mathbb{G}$ induces a slender splitting of $\mathbb{G}'$.*

For cyclic splittings, the conjecture is a theorem of Panos Papasoglu (Theorem 4.4); this is the key to our proof of Corollary 4.9. When $\mathbb{G}$ is also the fundamental group of a 3-manifold, then the conjecture is a theorem of Kapovich–Leeb [KL97, Thm. 1.1].

Using Lemma 3.5 and a proposition of Gilbert Levitt (Proposition 2.6), we give a group invariant characterisation of the polynomial part of a free-by-cyclic group using cyclic splittings. The characterisation also follows from Lemma 4.1 and relative hyperbolicity.

Proposition 3.6. *There is a unique (up to conjugacy) finite set $\mathcal{P}(\mathbb{G})$ of (pairwise) non-conjugate free-by-cyclic subgroups of $\mathbb{G}$ such that a free-by-cyclic subgroup of $\mathbb{G}$ is conjugate into some subgroup in $\mathcal{P}(\mathbb{G})$ if and only if it is polynomial.*

Bestvina–Feighn–Handel [BFH00, §3] associated a partially ordered finite set $\Lambda^+(\phi)$ of attracting laminations to any outer automorphism ϕ of $\mathbb{F}$; $\Lambda^+(\phi)$ is empty if and only if ϕ is polynomially growing. The height $\mathfrak{h}(\phi)$ is the length of the longest chains in $\Lambda^+(\phi)$.

Conjecture 5.5. *If $\mathbb{G}, \mathbb{G}'$ are quasi-isometric, then $\mathfrak{h}(\phi) = \mathfrak{h}(\psi)$.*

It is open whether height is even a group invariant! This conjecture is closely related to our previous conjecture that being both fully irreducibly and atoroidal was a geometric invariant – see the discussion at the end of Section 5.

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2 Polynomial growth

Throughout, $\mathbb{F}$ denotes a nontrivial finitely generated free group, $\Phi\colon \mathbb{F} \to \mathbb{F}$ an automorphism, and $\phi = [\Phi]$ its outer class.

A *tree* is a contractible 1-dimensional cell complex. A cyclic splitting of a group is a simplicial tree (i.e. contractible graph) with a minimal nontrivial simplicial action of the group whose edge stabilisers are cyclic; it is a free splitting if the edge stabilisers are trivial. A free splitting of $\mathbb{F}$ is absolute if vertex stabilisers are trivial – this means the action is free, but we introduce "absolute" to avoid the phrase "free free splittings" later on.

Let T be an absolute free splitting of $\mathbb{F}$. For $x \in \mathbb{F}$, the *T-length* $\|x\|_T$ is the minimal distance x translates a point in T. A conjugacy class $[x]$ *grows exponentially* on (forward) ϕ-iteration if $\liminf_{n\to\infty} \|\Phi^n(x)\|_T^{1/n} > 1$; it *grows polynomially* on ϕ-iteration if the sequence $(\|\Phi^n(x)\|_T)_{n=1}^{\infty}$ is bounded above by a polynomial in n — the smallest degree of such a polynomial is the *degree for* $[x]$. These properties (grows exponentially/polynomially & limit-inferior/degree) are independent of the absolute free splitting T. For the interested reader, Levitt gives a finer classification of growth rates in [Lev09, §6]. The outer automorphism ϕ is exponentially growing if some conjugacy class $[x]$ grows exponentially on ϕ-iteration; it is polynomially growing if every conjugacy class $[x]$ grows polynomially on ϕ-iteration.

A self-map $g\colon T \to T$ of a cyclic splitting of $\mathbb{F}$ is Φ-equivariant if $\widehat{g}(x \cdot p) = \Phi(x) \cdot \widehat{g}(p)$ for all $x \in \mathbb{F}, p \in \widehat{\Gamma}_G$. A cyclic splitting T of $\mathbb{F}$ is ϕ-fixed if it admits a Φ-equivariant simplicial automorphism; such a simplicial automorphism is unique when $\mathbb{F}$ is not cyclic. Marc Culler [Cul84, Thm. 2.1] proved our first characterisation of a dynamical property of ϕ in terms of absolute free splittings:

Theorem 2.1. *$\mathbb{F}$ has a ϕ-fixed absolute free splitting if and only if ϕ has finite order.* □

By solving the Nielsen realisation problem for $\mathrm{Out}(\mathbb{F})$, Culler actually proved a stronger theorem: $\mathfrak{G} \le \mathrm{Out}(\mathbb{F})$ is finite if and only if $\mathbb{F}$ has a $\mathfrak{G}$-*fixed* absolute free splitting. Bestvina–Handel [BH92, §1] generalised Theorem 2.1 to polynomial growth using the next crucial observation:

Theorem 2.2. *If $\mathbb{F}$ has no ϕ-fixed free splitting, then ϕ is exponentially growing.* □

Let T be a cyclic splitting of $\mathbb{F}$. By Bass–Serre theory [Ser77, I-§5.4], there are finitely many orbits $\mathbb{F} \cdot v_i$ in T of vertices with nontrivial stabilisers, and the conjugacy classes $[\mathbb{S}_i]$ of nontrivial vertex stabilisers are represented by finitely generated proper subgroups of $\mathbb{F}$; we will refer to the representatives $\mathbb{S}_i$ as the *children* of T. Any Φ-equivariant simplicial automorphism $g\colon T \to T$ permutes the orbits $\mathbb{F} \cdot v_i$ of vertices with nontrivial stabilisers, the conjugacy classes $[\mathbb{S}_i]$ of the children are ϕ-periodic, and the restrictions $\phi_i \in \mathrm{Out}(\mathbb{S}_i)$ are the outer automorphisms induced by the first-return maps $g_i\colon \mathbb{F} \cdot v_i \to \mathbb{F} \cdot v_i$ under g; ϕ is exponentially growing if some restriction ϕ_i is exponentially growing.

Now assume T is a ϕ-fixed free splitting. So the children are proper free factors $\mathbb{F}_i$; in particular, they have smaller rank than $\mathbb{F}$, and we can induct on rank. By inductively considering ϕ_i-fixed free splittings of the children $\mathbb{F}_i$, we get a ϕ-fixed (free) hierarchy for $\mathbb{F}$: a *family tree* starting with $\mathbb{F}$ such that each terminal *descendant* $\mathbb{F}'$ has a ψ-fixed absolute free splitting or no ψ-fixed free splitting, where $\psi \in \mathrm{Out}(\mathbb{F}')$ is the restriction of ϕ; a ϕ-fixed hierarchy is complete if its terminal descendants have fixed absolute free splittings. The *depth* of a ϕ-fixed hierarchy is the length of the longest branches in the family tree. The depth $\delta(\phi)$ of ϕ is the minimal depth of ϕ-fixed hierarchies for $\mathbb{F}$; for instance, $\delta(\phi) = 0$ means $\mathbb{F}$ has a ϕ-fixed absolute free splitting or no ϕ-fixed free splitting.

We use ϕ-fixed hierarchies for $\mathbb{F}$ to prove the dichotomy of growth types:

Proposition 2.3. *The following are equivalent:*

1. *every ϕ-fixed hierarchy for $\mathbb{F}$ is complete;*
2. *some ϕ-fixed hierarchy for $\mathbb{F}$ is complete;*

3. ϕ *is polynomially growing; and*

4. ϕ *is not exponentially growing.*

Sketch. (1⇒2) is immediate since $\mathbb{F}$ has at least one ϕ-fixed hierarchy.

2⇒3: Suppose $\mathbb{F}$ has a complete ϕ-fixed hierarchy with depth d and let $g\colon T \to T$ denote the initial Φ-equivariant simplicial automorphism of a free splitting that produced it. In the base case, T is absolute and the length of every edge-path in T is preserved on g-iteration, i.e. grows polynomially on g-iteration with degree $0 = d$.

For induction, assume T_i are absolute free splittings of $\mathbb{F}_i$, the children of T, and they admit Φ_i-equivariant self-maps $g_i\colon T_i \to T_i$ for which every edge path in T_i *grows polynomially (rel. endpoints)* with degree $< d$ on g_i-iteration. Let T' be a *blow up* of T rel. T_i, i.e. an absolute free splitting of $\mathbb{F}$ with $\mathbb{F}_i$-equivariant copies of T_i. The self-maps g and g_i induce a Φ-equivariant self-map $g'\colon T' \to T'$ for which every edge-path in T' grows polynomially with degree $\leq d$ on g'-iteration. By induction, the conjugacy class $[x]$ grows polynomially with degree $\leq d$ on ϕ-iteration for all $x \in \mathbb{F}$.

3⇒4: Polynomial functions are subexponential.

4⇒1: Suppose some ϕ-fixed hierarchy for $\mathbb{F}$ is not complete, i.e. some terminal descendant $\mathbb{F}' \leq \mathbb{F}$ has no ψ-fixed free splitting, where $\psi \in \mathrm{Out}(\mathbb{F}')$ is the restriction of ϕ. So ψ, and hence ϕ, is exponentially growing by Theorem 2.2. □

Let ϕ be polynomially growing. The proof shows that the degree for $[x]$ is at most $\delta(\phi)$ for all $x \in \mathbb{F}$; define <u>degree</u> $\deg(\phi)$ of ϕ to be the maximum such degree. A ϕ-fixed cyclic splitting of $\mathbb{F}$ is <u>simplifying</u> if it determines restrictions ϕ_i with $\deg(\phi_i) < \deg(\phi)$. The next strengthening of Theorem 2.2 is due to Bestvina–Feighn–Handel [BFH05, §4]:

Theorem 2.4. $\mathbb{F}$ *has a simplifying* ϕ*-fixed cyclic splitting when* ϕ *is polynomially growing.*

Sketch. If $\deg(\phi) = 0$, then ϕ has finite order and, by Theorem 2.1, $\mathbb{F}$ has a ϕ-fixed absolute free splitting. If $\deg(\phi) = 1$, then the ϕ-fixed cyclic splitting in [BFH05, Lem. 4.37] determines finite order restrictions ϕ_i. Finally, if $\deg(\phi) \geq 2$, then the ϕ-fixed free splitting in [BFH05, Lem. 4.33] determines restrictions ϕ_i with $\deg(\phi_i) < \deg(\phi)$. □

A <u>ϕ-fixed cyclic hierarchy</u> for $\mathbb{F}$ is a family tree that starts with $\mathbb{F}$, consists of descendants that are children of fixed cyclic splittings, and terminates on descendants that have fixed absolute free splittings or no fixed cyclic splittings. Let ϕ be polynomially growing. By inductively picking simplifying ϕ_i-fixed cyclic splittings of the children $\mathbb{S}_i$, we get a ϕ-fixed cyclic hierarchy for $\mathbb{F}$ whose depth is $\deg(\phi)$. As in the proof of Proposition 2.3, the degree $\deg(\phi)$ is at most the depth of any ϕ-fixed cyclic hierarchy for $\mathbb{F}$. This characterises $\deg(\phi)$ as the minimal depth of ϕ-fixed cyclic hierarchies for $\mathbb{F}$.

Corollary 2.5. *If* ϕ *is polynomially growing, then so is* ϕ^{-1} *and* $\deg(\phi^{-1}) = \deg(\phi)$.

Proof. Let T be a free splitting of $\mathbb{F}$ and $g\colon T \to T$ a Φ-equivariant simplicial automorphism. Then the inverse $g^{-1}\colon T \to T$ is a Φ^{-1}-equivariant simplicial automorphism; thus T is also ϕ^{-1}-fixed. So ϕ-fixed hierarchies for $\mathbb{F}$ are also ϕ^{-1}-fixed, and ϕ^{-1} is polynomially growing by Proposition 2.3. Similarly, ϕ-fixed cyclic hierarchies for $\mathbb{F}$ are also ϕ^{-1}-fixed. □

When ϕ is exponentially growing, Levitt [Lev09, Prop. 1.4] remarkably showed that the polynomially growing conjugacy classes are supported in a unique subgroup system:

Proposition 2.6. *There is a finite set $\mathcal{P}(\phi)$ of (pairwise) nonconjugate nontrivial subgroups of $\mathbb{F}$ such that a nontrivial $x \in \mathbb{F}$ is conjugate to an element of a subgroup in $\mathcal{P}(\phi)$ if and only if $[x]$ grows polynomially on ϕ-iteration.*

Each $\mathbb{P} \in \mathcal{P}(\phi)$ is finitely generated. Essentially distinct *conjugates of $\mathbb{P}_1, \mathbb{P}_2 \in \mathcal{P}(\phi)$ have trivial intersection. Any nontrivial subgroup of $\mathbb{F}$ consisting entirely of elements whose conjugacy classes in $\mathbb{F}$ grow polynomially on ϕ-iteration is conjugate to a subgroup of some $\mathbb{P} \in \mathcal{P}(\phi)$; therefore, the finite set $\mathcal{P}(\phi)$ is unique (up to conjugacy).*

Conjugates $g_1\mathbb{P}_1g_1^{-1}, g_2\mathbb{P}_2g_2^{-1}$ are *essentially distinct* if $\mathbb{P}_1 \neq \mathbb{P}_2$ or $g_2^{-1}g_1 \notin \mathbb{P}_1 = \mathbb{P}_2$. The set $\mathcal{P}(\phi)$ can be empty, and $\mathcal{P}(\phi) = \{\mathbb{F}\}$ if and only if ϕ is polynomially growing; moreover, $\mathcal{P}(\phi^n) = \mathcal{P}(\phi)$ (up to conjugacy) for all $n \geq 1$.

Sketch. Suppose ϕ is exponentially growing. A sort of converse of Theorem 2.2 states that $\mathbb{F}$ has a minimal nontrivial isometric action $\alpha\colon \mathbb{F} \to \mathrm{Isom}(T)$ on an $\mathbb{R}$-*tree* T with trivial *arc stabilisers*, and the new action $\alpha \circ \Phi$ is equivariantly homothetic but not isometric to α. By Gaboriau–Levitt's index theory [GL95, Thm. III.2], nontrivial point stabilisers are finitely generated and partitioned into finitely many conjugacy classes permuted by ϕ; moreover, if $x \in \mathbb{F}$ does not fix a point in T, then $[x]$ grows exponentially on ϕ-iteration.

Each child $\mathbb{S}_i$ of T (i.e. a nontrivial point stabiliser representing a conjugacy class) has the *restriction* $\phi_i \in \mathrm{Out}(\mathbb{S}_i)$. By inductively considering children with exponentially growing restrictions, we get a family tree whose terminal descendants $\mathcal{P}(\phi)$ do not have exponentially growing restrictions. So the restriction to each terminal descendant $\mathbb{P} \in \mathcal{P}(\phi)$ is polynomially growing (Proposition 2.3). By construction of the family tree and the last statement of the previous paragraph, a nontrivial $x \in \mathbb{F}$ is conjugate to an element of some terminal descendant if and only if $[x]$ grows polynomially on ϕ-iteration.

For uniqueness of $\mathcal{P}(\phi)$, suppose a nontrivial subgroup $\mathbb{P}' \leq \mathbb{F}$ consists entirely of elements whose conjugacy classes in $\mathbb{F}$ grow polynomially on ϕ-iteration. By construction of the family tree, $\mathbb{P}'$ is conjugate to a subgroup of some unique terminal descendant. □

A corollary of the sketch is the growth type dichotomy for conjugacy classes: $[x]$ grows polynomially on ϕ-iteration if and only if it does not grow exponentially on ϕ-iteration. By uniqueness, the outer automorphism ϕ permutes the conjugacy classes of the subgroups in $\mathcal{P}(\phi)$. Each $\mathbb{P}_i \in \mathcal{P}(\phi)$ has the polynomially growing *restriction* $\psi_i \in \mathrm{Out}(\mathbb{P}_i)$ induced by the *first-return map* under ϕ. By Corollary 2.5 and uniqueness, $\mathcal{P}(\phi^{-1}) = \mathcal{P}(\phi)$.

3 Group invariance

Thoughout, $\mathbb{G}$ denotes the free-by-cyclic group $\mathbb{F}\rtimes_\Phi\mathbb{Z}$. Our goal is to give a characterisation of polynomial growth for ϕ as an algebraic property of $\mathbb{G}$.

Convention. $\mathbb{G}$ has a relative presentation $\langle \mathbb{F}, t \mid txt^{-1} = \Phi(x) \text{ for all } x \in \mathbb{F}\rangle$.

A slender splitting of a group is a simplicial tree with a minimal nontrivial simplicial action of the group whose edge stabilisers are slender.

Lemma 3.1. *If $\mathbb{F}$ is not cyclic, then slender (resp. cyclic) splittings of $\mathbb{G}$ are naturally in one-to-one correspondence with ϕ-fixed cyclic (resp. free) splittings of $\mathbb{F}$.*

A variation of this appears in the proof of [KK00, Cor. 15]. The variation is also proven in the second paragraph of [Bri02, §1].

Proof. Suppose T is a ϕ-fixed cyclic splitting of $\mathbb{F}$. As $\mathbb{F}$ is not cyclic, a unique simplicial automorphism $g\colon T \to T$ is Φ-equivariant, which is precisely the property needed to extend the $\mathbb{F}$-action on T to a $\mathbb{G}$-action by $t \cdot p = g(p)$ for $p \in T$ − this is the unique $\mathbb{G}$-action extending the $\mathbb{F}$-action on T. For any edge e of T with $\mathbb{F}$-stabiliser $\langle y\rangle$, an iterate g^n fixes the orbit $\mathbb{F}\cdot e$, and the $\mathbb{G}$-stabiliser of e is $\langle y, xt^n\rangle$ for some $x \in \mathbb{F}$ and minimal $n \geq 1$. If y is not trivial, then $\langle y, xt^n\rangle \cong \langle y\rangle \rtimes \langle xt^n\rangle$ by Φ-equivariance of g; otherwise, $\langle xt^n\rangle \cong \mathbb{Z}$. So T is a slender splitting of $\mathbb{G}$, and it is a cyclic splitting of $\mathbb{G}$ if T was a free splitting of $\mathbb{F}$.

Conversely, suppose T is a slender splitting of $\mathbb{G}$. As $\mathbb{F}$ is not cyclic, $\mathbb{F} \trianglelefteq \mathbb{G}$ acts nontrivially and minimally on T, and the action on T by $t \in \mathbb{G}$ is Φ-equivariant. So T is a ϕ-fixed slender (hence cyclic) splitting of $\mathbb{F}$. Now suppose T was a cyclic splitting of $\mathbb{G}$. Since $\mathbb{F}$ is finitely generated, T has finitely many $\mathbb{F}$-orbits of edges. For any edge e of T with a nontrivial $\mathbb{F}$-stabiliser $\langle y\rangle$, a power t^n fixes the orbit $\mathbb{F}\cdot e$, and the $\mathbb{G}$-stabiliser of e has the noncyclic subgroup $\langle y, xt^n\rangle$ for some $x \in \mathbb{F}$ and $n \geq 1$ — yet T is a cyclic splitting of $\mathbb{G}$! So T is a ϕ-fixed free splitting of $\mathbb{F}$. □

A cyclic splitting of $\mathbb{G}$ is absolute if its vertex stabilisers are cyclic. The *children* of a slender splitting of $\mathbb{G}$ are the representatives of conjugacy classes of noncyclic vertex stabilisers. A $\mathbb{Z}$-splitting of a group is a cyclic splitting whose edge stabilisers are infinite; we say a group "splits over $\mathbb{Z}$" if it has a $\mathbb{Z}$-splitting. A $\mathbb{Z}$-hierarchy for $\mathbb{G}$ is a family tree that starts with $\mathbb{G}$, consists of descendants that are children of $\mathbb{Z}$-splittings, and terminates on descendants that have absolute $\mathbb{Z}$-splittings or no $\mathbb{Z}$-splittings; a $\mathbb{Z}$-hierarchy is complete if its terminal descendants have absolute $\mathbb{Z}$-splittings.

Suppose $\mathbb{F}$ is cyclic, i.e. $\mathbb{G} \cong \mathbb{Z}\rtimes\mathbb{Z}$. A torsion-free group with a free splitting will contain a noncyclic free subgroup. As $\mathbb{G}$ is virtually abelian, it has no free splittings. So any cyclic splitting of $\mathbb{G}$ is a $\mathbb{Z}$-splitting. Since $\mathbb{F}$ is cyclic, it has a unique cyclic splitting, and it is an absolute free splitting; however, $\mathbb{G}$ has infinitely many (resp. exactly two) slender splittings if $\Phi\colon \mathbb{F} \to \mathbb{F}$ is the identity (resp. the involution), and they are all absolute $\mathbb{Z}$-splittings. In this case, $\mathbb{G}$ has a unique $\mathbb{Z}$-hierarchy, and it is complete.

Now suppose $\mathbb{F}$ is not cyclic. It follows from the proof of Lemma 3.1 that any slender splitting of $\mathbb{G}$ has infinite edge stabilisers, and its children are free-by-cyclic subgroups. By Lemma 3.1, $\mathbb{Z}$-hierarchies for $\mathbb{G}$ are naturally in one-to-one correspondence with ϕ-fixed hierarchies for $\mathbb{F}$; a $\mathbb{Z}$-hierarchy is complete exactly when it corresponds to a complete ϕ-fixed hierarchy for $\mathbb{F}$. Together with the previous paragraph, Proposition 2.3 translates into the group invariance of growth type:

Theorem 3.2. *The following are equivalent:*

1. *every $\mathbb{Z}$-hierarchy for $\mathbb{G}$ is complete;*
2. *some $\mathbb{Z}$-hierarchy for $\mathbb{G}$ is complete; and*
3. *ϕ is polynomially growing.* □

$\mathbb{G}$ is polynomial if the conditions of the theorem hold. It turns out that the degree of a polynomially growing outer automorphism is a group invariant too. A slender hierarchy for $\mathbb{G}$ is a family tree that starts with $\mathbb{G}$, consists of descendants that are children of slender splittings, and terminates on descendants that have absolute $\mathbb{Z}$-splittings or no slender splittings. The slender depth $\delta_s(\mathbb{G})$ of $\mathbb{G}$ is the minimal depth of slender hierarchies for $\mathbb{G}$. Lemma 3.1 translates our previous characterisation of $\deg(\phi)$ into the group invariant $\delta_s(\mathbb{G})$:

Corollary 3.3. $\delta_s(\mathbb{G}) = \deg(\phi)$ *when $\mathbb{G}$ is polynomial.* □

This characterisation of $\deg(\phi)$ may be known to experts but is not stated in the literature.

Suppose $\mathbb{G}' \leq \mathbb{G}$ is a finite index subgroup. Then $\mathbb{F}' = \mathbb{G}' \cap \mathbb{F}$ has finite index in $\mathbb{F}$; in particular, it is finitely generated. As $\mathbb{G}'$ is not a subgroup of $\mathbb{F}$, it is generated by $\mathbb{F}'$ and xt^n for some $x \in \mathbb{F}$ and $n \geq 1$. So $\mathbb{G}' = \mathbb{F}' \rtimes_\Psi \mathbb{Z}$ is a free-by-cyclic group, where $\Psi\colon \mathbb{F}' \to \mathbb{F}'$ be given by $x' \mapsto x\Phi^n(x')x^{-1}$. An immediate consequence of group invariance is *commensurability* invariance:

Corollary 3.4. *Let $\mathbb{G}' \leq \mathbb{G}$ be a finite index subgroup. $\mathbb{G}'$ is polynomial if and only if $\mathbb{G}$ is polynomial; $\delta_s(\mathbb{G}') = \delta_s(\mathbb{G})$ when $\mathbb{G}$ is polynomial.*

We suspect the identity $\delta_s(\mathbb{G}') = \delta_s(\mathbb{G})$ still holds without the polynomial assumption.

Proof. Since $\delta_s(\mathbb{G}) = \deg(\phi)$ when $\mathbb{G}$ is polynomial (Corollary 3.3), it suffices to show that growth type is inherited by proper (positive) powers/roots, restrictions to finite index subgroups, and extensions to finite index overgroups. For $x \in \mathbb{F}$, if the conjugacy class $[x]$ grows polynomially growing on ϕ-iteration with degree d, then it is immediate from the definition that $[x]$ grows polynomially on ϕ^n-iteration with degree d for $n \geq 1$. So ϕ^n is polynomially growing with $\deg(\phi^n) = \deg(\phi)$ if ϕ is polynomially growing for $n \geq 1$. Similarly, if $[x]$ grows exponentially on ϕ-iteration, then it also grows exponentially on ϕ^n-iteration for $n \geq 1$; thus ϕ^n is exponentially growing if ϕ is exponentially growing.

Suppose $\mathbb{F}' \leq \mathbb{F}$ be a finite index subgroup and $\Psi = \Phi^n|\mathbb{F}'$ for some $n \geq 1$. Pick an absolute free splitting Γ of $\mathbb{F}$; the finite cover Γ' corresponding to $\mathbb{F}'$ is an absolute free splitting of $\mathbb{F}'$. Then $\|x'\|_{\Gamma'} = \|x'\|_{\Gamma}$ for $x' \in \mathbb{F}'$, and the conjugacy class $[x']$ (in $\mathbb{F}'$) grows polynomially on ψ-iteration with degree d if and only if the conjugacy class $[x']$ (in $\mathbb{F}$) grows polynomially on ϕ^n-iteration with degree d. For this reason, ϕ^n is exponentially growing if ψ is exponentially growing. Conversely, if ψ is polynomially growing and $x \in \mathbb{F}$, then $x^m \in \mathbb{F}'$ for some $m \geq 1$ and the conjugacy class $[x^m]$ (and hence $[x]$) grows polynomially on ϕ^n-iteration with the same degree as on ψ-iteration; therefore, ϕ^n is polynomially growing, and $\deg(\psi) = \deg(\phi^n) = \deg(\phi)$. □

We now turn the subgroup system $\mathbb{P}(\phi)$ into a group invariant of $\mathbb{G}$. Each $\mathbb{P}_i \in \mathcal{P}(\phi)$ representing a ϕ-orbit of conjugacy classes determines a polynomial free-by-cyclic subgroup $\mathbb{G}_i = \mathbb{P}_i \rtimes_{\Phi_i} \mathbb{Z}$ of $\mathbb{G}$, where $\Phi_i \colon \mathbb{P}_i \to \mathbb{P}_i$ represents the polynomially growing restriction ϕ_i; denote the set of subgroups $\mathbb{G}_i$ by $\mathcal{P}(\mathbb{G})$. As before, $\mathcal{P}(\mathbb{G})$ can be empty, and $\mathcal{P}(\mathbb{G}) = \{\mathbb{G}\}$ if and only if $\mathbb{G}$ is polynomial. Our goal is to give an algebraic characterisation of $\mathcal{P}(\mathbb{G})$.

First, here is a surprising property of free-by-cyclic subgroups in $\mathbb{G}$:

Lemma 3.5. *If $\mathbb{G}' \leq \mathbb{G}$ is a free-by-cyclic subgroup, then $\mathbb{G}' \cap \mathbb{F}$ is finitely generated.*

This lemma was the key idea in [Mut21, Thm. 4.3]. It is indispensable for turning algebraic statements on $\mathbb{G}$ into dynamical statements on ϕ; despite its importance, it is not widely known. A minor variation is stated in the paragraph following [HW10, Thm. A].

Proof. Let $\pi \colon \mathbb{G} \to \mathbb{Z}$ be the homomorphism that maps $\mathbb{F} \mapsto 0$ and $t \mapsto 1$. Then $\mathbb{F}' = \mathbb{G}' \cap \mathbb{F}$ is the kernel of the restriction $\pi|\mathbb{G}'$. By our definition of free-by-cyclic groups, $\mathbb{G}'$ is a finitely generated noncyclic subgroup with Euler characteristic $\chi(\mathbb{G}') = 0$; thus $\mathbb{G}'$ is generated by $\mathbb{F}'$ and $s = xt^n$ for some $x \in \mathbb{F}$ and $n \geq 1$. In proving the coherence of $\mathbb{G} = \mathbb{F} \rtimes_{\Phi} \mathbb{Z}$, Feighn–Handel [FH99, Prop. 2.3] show that $\mathbb{G}'$ has the relative presentation

$$\langle \mathbb{F}'', s \mid sas^{-1} = x\Phi^n(a)x^{-1} \text{ for all } a \in \mathbb{A} \rangle,$$

where $\mathbb{A}$ is a free factor of a finitely generated subgroup $\mathbb{F}'' \leq \mathbb{F}'$ such that $x\Phi^n(\mathbb{A})x^{-1} \leq \mathbb{F}''$. This presentation is *aspherical* and allows us to compute: $0 = \chi(\mathbb{G}') = \operatorname{rank}(\mathbb{A}) - \operatorname{rank}(\mathbb{F}'')$; therefore, $\mathbb{A} = \mathbb{F}''$. Since $\pi|\mathbb{G}'$ maps $\mathbb{F}'' \mapsto 0$ and $s \mapsto n$ and its kernel $\mathbb{F}'$ is free, we have $x\Phi^n(\mathbb{F}'')x^{-1} = \mathbb{F}''$ — otherwise, if $x\Phi^n(\mathbb{F}'')x^{-1} \neq \mathbb{F}''$, then the kernel is (locally free but) not free. In particular, $\mathbb{F}' = \ker(\pi|\mathbb{G}') = \mathbb{F}''$ is finitely generated. □

We are now ready to give our algebraic characterisation:

Proposition 3.6. *$\mathcal{P}(\mathbb{G})$ is the unique (up to conjugacy) finite set of (pairwise) nonconjugate free-by-cyclic subgroups of $\mathbb{G}$ such that a free-by-cyclic subgroup of $\mathbb{G}$ is conjugate into some subgroup in $\mathcal{P}(\mathbb{G})$ if and only if it is polynomial.*

Although we do not prove (nor need) it here, $\mathcal{P}(\mathbb{G})$ is *malnormal*: essentially distinct conjugates of $\mathbb{G}_1, \mathbb{G}_2 \in \mathcal{P}(\mathbb{G})$ have trivial intersection. Another proof is given after Theorem 5.2.

Proof. Let $\mathbb{G}' \leq \mathbb{G}$ be a free-by-cyclic subgroup. By Lemma 3.5, $\mathbb{F}' = \mathbb{G}' \cap \mathbb{F}$ is finitely generated, and $\mathbb{G}'$ is generated by $\mathbb{F}'$ and $s \in \mathbb{G}$. In particular, $\mathbb{G}' = \mathbb{F}' \rtimes_\Psi \mathbb{Z}$, where $\Psi\colon \mathbb{F}' \to \mathbb{F}'$ represents a restriction $\psi \in \mathrm{Out}(\mathbb{F}')$. If $xt^n \in \mathbb{G}$ conjugates $\mathbb{G}'$ into some subgroup in $\mathcal{P}(\mathbb{G})$ for some $x \in \mathbb{F}, n \in \mathbb{Z}$, then x conjugates $\Phi^n(\mathbb{F}')$ into a subgroup in $\mathcal{P}(\phi)$. This implies $[x']$ grows polynomially on ϕ/ψ-iteration for all $x' \in \mathbb{F}'$, and $\mathbb{G}'$ is polynomial. Conversely, if $\mathbb{G}'$ is polynomial, then $[x']$ grows polynomially on ψ/ϕ-iteration for all $x' \in \mathbb{F}'$. By Proposition 2.6, $\mathbb{F}'$ is a subgroup of some $\mathbb{P}_i \in \mathcal{P}(\phi)$ (up to conjugacy in $\mathbb{F}$). Then s normalises $\mathbb{P}_i$, and $\mathbb{G}' = \langle \mathbb{F}', s\rangle$ is a subgroup of $\langle \mathbb{P}_i, s\rangle \leq \mathbb{G}_i \in \mathcal{P}(\mathbb{G})$. □

4 Geometric invariance

Throughout, $\mathbb{F}'$ denotes a finitely generated noncyclic free group, $\Psi\colon \mathbb{F}' \to \mathbb{F}'$ an automorphism, $\psi = [\Psi]$ its outer class, and $\mathbb{G}'$ the free-by-cyclic group $\mathbb{F}' \rtimes_\Psi \mathbb{Z}$. We also assume $\mathbb{G}$ is not virtually abelian, i.e. $\mathbb{F}$ is not cyclic.

The *Cayley graph* $\mathrm{Cay}(\mathbb{H}, S)$ of a group $\mathbb{H}$ with respect to a finite generating set $S \subset \mathbb{H}$ is a graph whose 0-skeleton is $\mathbb{H}$ and 1-cells connect $g, gs \in \mathbb{H}$ for all $(g, s) \in \mathbb{H} \times S$. A function $f\colon \Gamma \to \Gamma'$ on connected locally finite graphs is a quasi-isometric (q.i.) embedding if there is a constant $K \geq 1$ such that

$$\frac{1}{K} d(p_1, p_2) - K \leq d'(f(p_1), f(p_2)) \leq K d(p_1, p_2) + K \text{ for all } p_1, p_2 \in \Gamma,$$

where d, d' are the combinatorial metrics on Γ, Γ' respectively; f is a quasi-isometry if, additionally, Γ' is the K-neighbourhood of the image $f(\Gamma)$. Quasi-isometries determine an equivalence relation on connected locally finite graphs. For any two finite generating sets $S_1, S_2 \subset \mathbb{H}$, the identity map on $\mathbb{H}$ extends to a quasi-isometry $\mathrm{Cay}(\mathbb{H}, S_1) \to \mathrm{Cay}(\mathbb{H}, S_2)$; thus, $\mathrm{Cay}(\mathbb{H})$ is well-defined up to quasi-isometry. A finitely generated subgroup $\mathbb{H}' \leq \mathbb{H}$ is undistorted if the inclusion extends to a q.i.-embedding $\mathrm{Cay}(\mathbb{H}') \to \mathrm{Cay}(\mathbb{H})$; for instance, if $\mathbb{H}' \leq \mathbb{H}$ has finite index, then $\mathbb{H}'$ acts (by left multiplication) freely and cocompactly on $\mathrm{Cay}(\mathbb{H})$, and the inclusion $\mathbb{H}' \leq \mathbb{H}$ is a quasi-isometry.

Here is our second surprising property of free-by-cyclic subgroups in $\mathbb{G}$:

Lemma 4.1. *Free-by-cyclic subgroups of $\mathbb{G}$ are undistorted.*

Proof. To prove the lemma, it suffices to show that the free-by-cyclic subgroup $\mathbb{G}' \leq \mathbb{G}$ is undistorted in a finite index subgroup of $\mathbb{G}$. Lemma 3.5 states that $\mathbb{G}' \cap \mathbb{F}$ is finitely generated and we may assume $\mathbb{F}' = \mathbb{G}' \cap \mathbb{F}$. In particular, $\psi \in \mathrm{Out}(\mathbb{F}')$ is a restriction of an iterate of $\phi \in \mathrm{Out}(\mathbb{F})$. After replacing $\mathbb{G}$ with a finite index subgroup if necessary, we may assume ψ is a restriction of ϕ and $\mathbb{G}'$ is generated by $\mathbb{F}'$ and t (i.e. $\Psi = \Phi|\mathbb{F}'$). By Hall's theorem, $\mathbb{F}'$ is a free factor of a finite index subgroup of $\mathbb{F}$ (see [Sta83, §6]); we may replace $\mathbb{F}$ with the intersection of the Φ-iterates of the finite index subgroup and assume $\mathbb{F}' \leq \mathbb{F}$ is a free factor. Pick a basis $\mathfrak{B}'$ of $\mathbb{F}'$ and extend it to a basis $\mathfrak{B}$ of $\mathbb{F}$. We will show that the inclusion of $\mathrm{Cay}(\mathbb{G}') = \mathrm{Cay}(\mathbb{G}', \mathfrak{B}' \cup \{t\})$ into $\mathrm{Cay}(\mathbb{G}) = \mathrm{Cay}(\mathbb{G}, \mathfrak{B} \cup \{t\})$ is a q.i.-embedding by defining a map $r\colon \mathrm{Cay}(\mathbb{G}) \to \mathrm{Cay}(\mathbb{G}')$ that is a Lipschitz retract.

Note that $\mathrm{Cay}(\mathbb{F}') = \mathrm{Cay}(\mathbb{F}', \mathfrak{B}')$ is a subtree of $\mathrm{Cay}(\mathbb{F}) = \mathrm{Cay}(\mathbb{F}, \mathfrak{B})$ and the closest point projection $s\colon \mathrm{Cay}(\mathbb{F}) \to \mathrm{Cay}(\mathbb{F}')$ is 1-Lipschitz. Define r on the t^n-translates of $\mathrm{Cay}(\mathbb{F}) \subset \mathrm{Cay}(\mathbb{G})$ by setting $r(t^n \cdot p) = t^n \cdot s(p)$ for all $p \in \mathrm{Cay}(\mathbb{F}), n \in \mathbb{Z}$. For each remaining 1-cell e of $\mathrm{Cay}(\mathbb{G})$, let r map e linearly to an arbitrary shortest path in $\mathrm{Cay}(\mathbb{G}')$ connecting the r-images of its endpoints. The restriction $r|\,\mathrm{Cay}(\mathbb{G}')$ is the identity map.

We now exhibit a uniform diameter for the r-image of any 1-cell in $\mathrm{Cay}(\mathbb{F})$. This is immediate for 1-cells in $t^n \cdot \mathrm{Cay}(\mathbb{F})$ for $n \in \mathbb{Z}$ since s is 1-Lipschitz. Consider the 1-cell e connecting $t^{n-1}\Phi(x) = t^n x t^{-1}$ and $t^n x$, where $x \in \mathbb{F}$. We need a uniform bound on the distance in $\mathrm{Cay}(\mathbb{G}')$ between $r(t^{n-1}\Phi(x)) = t^{n-1}s(\Phi(x))$ and $r(t^n x) = t^n s(x)$. The latter is adjacent to $t^{n-1}\Phi(s(x))$, and it is enough to give a uniform bound on the distance in $\mathrm{Cay}(\mathbb{F}')$ between $s(\Phi(x))$ and $\Phi(s(x))$. Set $y = \Phi^{-1}(s(\Phi(x))) \in \mathbb{F}'$ and consider the shortest path $[x, y]$ in $\mathrm{Cay}(\mathbb{F})$ between x, y. Then $s(x) \in [x, y]$ and $\Phi(y) = s(\Phi(x)) \in [\Phi(x), \Phi(s(x))]$ by definition of s. Bounded cancellation [Coo87, p. 454] states that some uniform constant, independent of $x \in \mathbb{F}$, bounds the distance in $\mathrm{Cay}(\mathbb{F})$ between $\Phi(s(x))$ and $[\Phi(x), s(\Phi(x))]$, which is also the distance in $\mathrm{Cay}(\mathbb{F}')$ between $\Phi(s(x))$ and $s(\Phi(x))$. □

By a very similar argument, the lemma extends to cyclic subgroups.

Lemma 4.2. *Slender subgroups of* $\mathbb{G}$ *are undistorted.*

A variation of this appears in [Mit98, §3].

Proof. Any subgroup $\mathbb{Z} \rtimes \mathbb{Z} \le \mathbb{G}$ is undistorted by Lemma 4.1. First, suppose $\langle c \rangle \le \mathbb{G}$ is not a subgroup of $\mathbb{F}$. After replacing $\mathbb{G}$ with the finite index subgroup generated by $\mathbb{F}$ and c, we may assume $c = t$. Then $\langle c \rangle$ is a section of the homomorphism $\mathbb{G} \to \mathbb{Z}$ that maps $\mathbb{F}$ to 0 and t to 1; thus $\langle c \rangle \le \mathbb{G}$ is undistorted. Now suppose $\langle c \rangle \le \mathbb{F}$. If the conjugacy class $[c]$ (in $\mathbb{F}$) is ϕ-periodic, then $\langle c \rangle \le \mathbb{Z}^2 \le \mathbb{G}$. By Lemma 4.1 again, $\mathbb{Z}^2 \le \mathbb{G}$ is undistorted. All cyclic subgroups of $\mathbb{Z}^2$ are undistorted, and hence $\langle c \rangle \le \mathbb{G}$ is undistorted.

We may assume the conjugacy class $[c]$ *strictly grows* on $\phi^{\pm 1}$-iteration. Fix a basis for $\mathbb{F}$; for nontrivial $x \in \mathbb{F}$, let $\alpha(x) \subset \mathrm{Cay}(\mathbb{F})$ denote the axis for x in the Cayley tree. Consider $C = \bigcup_{n \in \mathbb{Z}} t^n \cdot \alpha(\Phi^{-n}(c))$ in $\mathrm{Cay}(\mathbb{G})$. Make C connected by including, for each vertex $t^{n+1}x \in C$, the edge connecting it to $t^n\Phi(x)$ and the shortest path in $t^n \cdot \mathrm{Cay}(\mathbb{F})$ from $t^n\Phi(x)$ to C; by bounded cancellation, the length of the latter paths are uniformly bounded. Let $s_n\colon \mathrm{Cay}(\mathbb{F}) \to \alpha(\Phi^{-n}(c))$ be the closest point projections, and define $r\colon \mathrm{Cay}(\mathbb{G}) \to C$ by setting $r(t^n \cdot p) = t^n \cdot s_n(p)$ for all $p \in \mathrm{Cay}(\mathbb{F}), n \in \mathbb{Z}$ and extending linearly on the remaining edges. As in the previous proof, C is undistorted in $\mathrm{Cay}(\mathbb{G})$ if there is a uniform bound on the distance in $\mathrm{Cay}(\mathbb{F})$ between $s_n(\Phi(x))$ and $\Phi(s_{n+1}(x))$ for $x \in \mathbb{F}$, which follows from bounded cancellation again. Since $[c]$ strictly grows on $\phi^{\pm 1}$-iteration, $\alpha(c)$ is *quasiconvex* in C; therefore, $\langle c \rangle \le \mathbb{G}$ is undistorted. □

A property of connected locally finite graphs is a geometric invariant if it is preserved by quasi-isometries; for instance, being one-ended (i.e. the complement of any bounded set has exactly one unbounded component) is a geometric invariant. Our goal in this section is to relate growth type of ϕ with a geometric invariant of $\mathrm{Cay}(\mathbb{G})$!

A connected subgraph $\ell \subset \mathrm{Cay}(\mathbb{G})$ is a quasi-geodesic if there is a q.i.-embedding $q\colon \mathrm{Cay}(\mathbb{Z}) \to \mathrm{Cay}(\mathbb{G})$ whose image is in ℓ and has a *finite neighbourhood* (i.e. the (closed) M-neighbourhood $N_M(\ell)$ for some $M \geq 0$) containing ℓ; a component of $\mathrm{Cay}(\mathbb{G}) \setminus \ell$ is *essential* if its union with ℓ is one-ended. A quasi-geodesic ℓ *(strongly) separates* $\mathrm{Cay}(\mathbb{G})$ if $\mathrm{Cay}(\mathbb{G}) \setminus \ell$ has at least two essential components and a function $D\colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that if $E \geq 0$ and C is a component of $\mathrm{Cay}(\mathbb{G}) \setminus N_E(\ell)$ that is not in $N_{D(E)}(\ell)$, then C is not in any finite neighbourhood of ℓ, and ℓ is in $N_{D(E)}(C)$. The existence of a separating quasi-geodesic is a geometric invariant:

Lemma 4.3. *If $f\colon \mathrm{Cay}(\mathbb{G}) \to \mathrm{Cay}(\mathbb{G}')$ is a quasi-isometry and ℓ a quasi-geodesic separating $\mathrm{Cay}(\mathbb{G})$, then some finite neighbourhood of $f(\ell)$ is a quasi-geodesic separating $\mathrm{Cay}(\mathbb{G}')$.*

This is stated in [Pap05, Lem. 1.7] but with a weaker notion of separating; however, that statement fails with the weaker condition. The stronger notion is due to Papasoglu too. The cited lemma dealt with *quasi-lines*, but quasi-geodesics will do for $\mathbb{G}$ thanks to Lemma 4.2.

Sketch. Let $K \geq 1$ be the *q.i.-constant* for f. Then the K-neighbourhood $N_K(f(\ell))$ is connected. Let $q\colon \mathrm{Cay}(\mathbb{Z}) \to \mathrm{Cay}(\mathbb{G})$ be a q.i.-embedding whose image is in ℓ and has a finite neighbourhood containing ℓ. Then $f \circ q$ is a q.i.-embedding whose image is in $f(\ell)$ and has a finite neighbourhood containing $N_K(f(\ell))$. So $\ell' = N_K(f(\ell))$ is a quasi-geodesic.

Suppose ℓ is separating with corresponding function $D\colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$. For $E' \geq 0$, set $D'(E') = \max(E'K^2 + 6K^3 + K^2 + K, D(E'K + 4K^2)K + 2K)$. Let C' be a component of $\mathrm{Cay}(\mathbb{G}') \setminus N_{E'}(\ell')$ that is not in $N_{D'(E')}(\ell')$. Then some component C'' of $C' \setminus N_{E'K^2+7K^3+K^2+2K}(f(\ell))$ is not in $N_{D(E'K+4K^2)K+K}(f(\ell))$. The preimage $f^{-1}(C'')$ is in a component C of $\mathrm{Cay}(\mathbb{G}) \setminus N_{E'K+4K^2}(\ell)$ but not in $N_{D(E'K+4K^2)}(\ell)$. So C is not in any finite neighbourhood of ℓ, and ℓ is in $N_{D(E'K+4K^2)}(C)$. Finally, f maps C into the component C' of $\mathrm{Cay}(\mathbb{G}') \setminus N_{E'+K}(f(\ell))$; thus C' is not in any finite neighbourhood of ℓ', and ℓ' is in $N_{D'(E')}(C')$.

Let $E' \geq 0$ and C' be a component of $\mathrm{Cay}(\mathbb{G}') \setminus N_{E'}(\ell')$ that is not in any finite neighbourhood of ℓ'. For a contradiction, suppose that the complement in $C' \cup N_{E'}(\ell')$ of some connected compact subgraph $K \subset C' \cup N_{E'}(\ell')$ has at least two unbounded components. Replace K with a larger connected compact subgroup such that $N_{E'}(\ell') \setminus K$ has exactly two components and they are unbounded, and $(C' \cup N_{E'}(\ell')) \setminus K$ has only unbounded components. As $\mathrm{Cay}(\mathbb{G}')$ is one-ended, each component of $(C' \cup N_{E'}(\ell')) \setminus K$ contains a component of $N_{E'}(\ell') \setminus K$. So $(C' \cup N_{E'}(\ell')) \setminus K$ has exactly two unbounded components $C'_\pm$, and no finite neighbourhood of C'_+ or C'_- contains ℓ'. Pick $R \geq 0$ so that $N_{E'+R}(\ell')$ contains K. Without loss of generality, some component of $C'_+ \setminus N_{E'+R}(\ell')$ is a component of $\mathrm{Cay}(\mathbb{G}') \setminus N_{E'+R}(\ell')$ that is not in any finite neighbourhood of ℓ'. By the previous paragraph, ℓ' is in some finite neighbourhood of C'_+ – a contradiction.

Let $C_1 \neq C_2$ be two essential components of $\mathrm{Cay}(\mathbb{G})\setminus\ell$. For $i = 1, 2$, some component C_i^* of $C_i \setminus N_{3K^4+2K^3+5K^2}(\ell)$ is not in any finite neighbourhood of ℓ. So f maps C_i^* into the component C'_i of $\mathrm{Cay}(\mathbb{G}') \setminus N_{3K^3+2K^2+2K}(f(\ell))$ that is not in any finite neighbourhood of $f(\ell)$. For a contradiction, suppose $C' = C'_1 = C'_2$. Then the preimage $f^{-1}(C')$ is in a component of $\mathrm{Cay}(\mathbb{G}) \setminus N_K(\ell)$. So C_1^*, C_2^* are in the same component of $\mathrm{Cay}(\mathbb{G}) \setminus \ell$, which contradicts $C_1 \neq C_2$; therefore, $C'_1 \neq C'_2$ and $N_{3K^3+2K^2+K}(\ell')$ is separating. □

The following is a deep theorem of Papasoglu [Pap05, Thm. 1]:

Theorem 4.4. $\mathrm{Cay}(\mathbb{G})$ *has a separating quasi-geodesic if and only if* $\mathbb{G}$ *splits over* $\mathbb{Z}$. □

So splitting over $\mathbb{Z}$ is a geometric invariant of $\mathbb{G}$! The cited theorem applies to more general finitely presented groups. We have specialised and simplified all statements in this section for our free-by-cyclic group $\mathbb{G}$. The following conjecture motivates an extension of Theorem 4.4 to slender splittings.

Conjecture 4.5. *If* $\mathbb{G}, \mathbb{G}'$ *are quasi-isometric, then a slender splitting of* $\mathbb{G}$ *induces a slender splitting of* $\mathbb{G}'$.

Let T, T' be $\mathbb{Z}$-splittings of $\mathbb{G}$. T dominates T' if every T-point stabiliser fixes a point in T'. Two $\mathbb{Z}$-splittings of $\mathbb{G}$ are equivalent if they dominate each other; domination induces a partial order on the equivalence classes. Note that equivalent $\mathbb{Z}$-splittings have the same children. Analogous to *JSJ-decompositions for 3-manifolds*, Rips–Sela [RS97, Thm. 7.1] defined a canonical equivalence class of $\mathbb{Z}$-splittings:

Theorem 4.6. *Some* $\mathbb{Z}$*-splitting of* $\mathbb{G}$ *dominates all* $\mathbb{Z}$*-splittings of* $\mathbb{G}$. □

This theorem (and what follows) uses our assumption $\mathbb{G}$ is not virtually abelian. The *(canonical)* $\mathbb{Z}$*-children* of $\mathbb{G}$ are the children of the maximal equivalence class of $\mathbb{Z}$-splittings of $\mathbb{G}$. By inductively considering the $\mathbb{Z}$*-descendants*, we get the canonical $\mathbb{Z}$-hierarchy for $\mathbb{G}$ whose depth is the $\mathbb{Z}$-depth $\delta_{\mathbb{Z}}(\mathbb{G})$ of $\mathbb{G}$. Note that $\delta_{\mathbb{Z}}(\mathbb{G}) = 0$ means $\mathbb{G}$ has no $\mathbb{Z}$-children, i.e. $\mathbb{G}$ has an absolute $\mathbb{Z}$-splitting or does not split over $\mathbb{Z}$.

Two subsets of a metric space are a finite Hausdorff distance apart if each subset is in the finite neighbourhood of the other. By another theorem of Papasoglu [Pap05, Thm. 7.1], the JSJ-children are geometric invariants:

Theorem 4.7. *If* $f\colon \mathrm{Cay}(\mathbb{G}) \to \mathrm{Cay}(\mathbb{G}')$ *is a quasi-isometry and* $\mathbb{G}_0$ *a* $\mathbb{Z}$*-child of* $\mathbb{G}$*, then* $f(\mathbb{G}_0)$ *is a finite Hausdorff distance from a* $\mathbb{Z}$*-child of* $\mathbb{G}'$. □

We say $\mathbb{G}, \mathbb{G}'$ are quasi-isometric if their Cayley graphs are quasi-isometric.

Corollary 4.8. *If* $\mathbb{G}, \mathbb{G}'$ *are quasi-isometric, then* $\delta_{\mathbb{Z}}(\mathbb{G}') = \delta_{\mathbb{Z}}(\mathbb{G})$*; furthermore, if the canonical* $\mathbb{Z}$*-hierarchy for* $\mathbb{G}$ *is complete, then so is the canonical* $\mathbb{Z}$*-hierarchy for* $\mathbb{G}'$.

Proof. By Theorem 4.7, $\mathbb{G}$ has a $\mathbb{Z}$-child (i.e. $\delta_{\mathbb{Z}}(\mathbb{G}) > 0$) if and only if $\mathbb{G}'$ does too. Thus the first part of theorem holds if $\delta_{\mathbb{Z}}(\mathbb{G}) = 0$. Suppose $\delta_{\mathbb{Z}}(\mathbb{G}) > 0$ and the first part holds for free-by-cyclic groups with $\mathbb{Z}$-depth $< \delta_{\mathbb{Z}}(\mathbb{G})$. By Theorem 4.7 and Lemma 4.1, each $\mathbb{Z}$-child $\mathbb{G}_i$ of $\mathbb{G}$ is quasi-isometric to a $\mathbb{Z}$-child $\mathbb{G}'_{j_i}$ of $\mathbb{G}'$, and $\delta_{\mathbb{Z}}(\mathbb{G}_i) < \delta_{\mathbb{Z}}(\mathbb{G})$. Similarly, each $\mathbb{Z}$-child $\mathbb{G}'_j$ of $\mathbb{G}'$ is quasi-isometric to a $\mathbb{Z}$-child $\mathbb{G}_{i_j}$ of $\mathbb{G}$. By the induction hypothesis, $\delta_{\mathbb{Z}}(\mathbb{G}_i) = \delta_{\mathbb{Z}}(\mathbb{G}'_{j_i})$, $\delta_{\mathbb{Z}}(\mathbb{G}'_j) = \delta_{\mathbb{Z}}(\mathbb{G}_{i_j})$ for all $\mathbb{Z}$-children of $\mathbb{G}, \mathbb{G}'$; therefore, $\delta_{\mathbb{Z}}(\mathbb{G}') = \delta_{\mathbb{Z}}(\mathbb{G})$.

For the second part of the theorem, assume the canonical $\mathbb{Z}$-hierarchy for $\mathbb{G}$ is complete and set $\delta = \delta_{\mathbb{Z}}(\mathbb{G}') = \delta_{\mathbb{Z}}(\mathbb{G})$. If $\delta = 0$, then both $\mathbb{G}, \mathbb{G}'$ have absolute $\mathbb{Z}$-splittings by Theorem 4.4 and Lemma 4.3, and the canonical $\mathbb{Z}$-hierarchy for $\mathbb{G}'$ is complete too. Suppose $\delta > 0$ and the second part of the theorem holds for free-by-cyclic groups with $\mathbb{Z}$-depth $< \delta$. As above, each $\mathbb{Z}$-child of $\mathbb{G}'$ is quasi-isometric to a $\mathbb{Z}$-child of $\mathbb{G}$, and the latter are assumed to have complete canonical $\mathbb{Z}$-hierarchies. By the induction hypothesis, each $\mathbb{Z}$-child of $\mathbb{G}'$ has a complete canonical $\mathbb{Z}$-hierarchy and so does $\mathbb{G}'$. □

By Theorem 3.2 and Corollary 4.8, growth type is a geometric invariant:

Corollary 4.9. *If $\mathbb{G}, \mathbb{G}'$ are quasi-isometric and $\mathbb{G}$ is polynomial, then so is $\mathbb{G}'$.* □

This also follows from Theorem 5.2 below. Macura [Mac02, Thm. 1.2] proved the geometric invariance of the degree of a polynomially growing outer automorphism:

Theorem 4.10. *A polynomial $\mathbb{G}$ has polynomial divergence with degree* $\deg(\phi) + 1$. □

Roughly speaking, the *divergence* is a (collection of) function(s) that measures how quickly geodesic rays diverge. The interested reader should refer to Stephen Gersten's paper [Ger94, §2], where divergence is introduced and noted to be a geometric invariant. Recently, Mark Hagen [Hag19, Thm. 1.2] 'modernised' Macura's theorem by showing that a polynomial $\mathbb{G}$ is *strongly thick* of order $\deg(\phi)$. *Thickness* is a more structural property (compared to divergence) that was introduced by Behrstock–Druţu–Mosher [BDM09, §7] as an obstruction to relative hyperbolicity.

Alternatively, one could reprove the geometric invariance of the degree by showing: $\delta_s(\mathbb{G})$ is the depth of a canonical slender hierarchy for $\mathbb{G}$; and this hierarchy is a geometric invariant. Fujiwara–Papasoglu [FP06, Thm. 5.13] already developed a slender analogue of Rips–Sela's Theorem 4.6. We currently have no geometric characterisation of when $\mathbb{G}$ splits over $\mathbb{Z} \rtimes \mathbb{Z}$, hence our Conjecture 4.5.

5 Relative hyperbolicity

We finally address the geometry of free-by-cyclic groups that are not polynomial.

First, consider the extreme case when $\mathcal{P}(\mathbb{G})$ is empty, or equivalently, $\mathcal{P}(\phi)$ is empty. By Proposition 2.3, polynomially growing outer automorphisms of $\mathbb{F}$ have periodic conjugacy classes of nontrivial elements in $\mathbb{F}$, e.g. nontrivial elements in terminal descendants of a complete fixed hierarchy; thus, $\mathcal{P}(\phi)$ is empty if and only if ϕ is atoroidal: there are no ϕ-periodic conjugacy classes of nontrivial elements in $\mathbb{F}$. We remark that $\mathcal{P}(\mathbb{G})$ is empty if and only if $\mathbb{G}$ has no free abelian subgroup of rank 2. Peter Brinkmann [Bri00, Thm. 1.1] proved that the absence of $\mathbb{Z}^2$ subgroups is a geometric invariant:

Theorem 5.1. *The following are equivalent:*

1. $\mathrm{Cay}(\mathbb{G})$ *is hyperbolic;*
2. $\mathbb{G}$ *has no* $\mathbb{Z}^2$ *subgroups; and*
3. ϕ *is atoroidal.* □

A locally finite graph is hyperbolic if there is a $\delta \geq 0$ such that the δ-neighbourhood of the union of two sides of any geodesic triangle in the graph contains the third side of the triangle; Misha Gromov [Gro87, Cor. 2.3.E] introduced this geometric invariant. Gromov also introduced in [Gro87, §8.6] the group property *relative hyperbolicity* that generalises non-uniform lattices of negatively curved symmetric spaces.

Theorem 5.1 gives a geometric characterisation of when $\mathcal{P}(\mathbb{G})$ is empty. Recently, this was generalised to the case when $\mathcal{P}(\mathbb{G}) \neq \{\mathbb{G}\}$, i.e. $\mathbb{G}$ is not polynomial.

Theorem 5.2. *The following are equivalent:*

1. $\mathbb{G}$ *is hyperbolic rel.* $\mathcal{P}(\mathbb{G})$*;*
2. $\mathbb{G}$ *is relatively hyperbolic;*
3. $\mathbb{G}$ *is not polynomial; and*
4. ϕ *is exponentially growing.*

Cornelia Druţu [Dru09, Thm. 1.2] proved that relative hyperbolicity was a geometric invariant! So this theorem gives another proof of Corollary 4.9.

Outline. (2⇒3) is essentially Macura's Theorem 4.10 since it is folklore that relatively hyperbolic groups have exponential divergence. Alternatively, this is Hagen's theorem since thick groups cannot be relatively hyperbolic.

(4⇒1) was initially announced by Gautero–Lustig in 2007, but their proof was incomplete. Pritam Ghosh [Gho23, Cor. 3.16] and Dahmani–Li [DL22, Thm. 0.4] recently gave complete independent proofs. □

Another proof for Proposition 3.6. If $\mathbb{G}$ is not polynomial, then $\mathbb{G}$ is hyperbolic rel. $\mathcal{P}(\mathbb{G})$ by Theorem 5.2. As any free-by-cyclic subgroup is undistorted (Lemma 4.1), $\mathbb{G}' \leq \mathbb{G}$ is either relatively hyperbolic or conjugate into some subgroup in $\mathcal{P}(\mathbb{G})$ [DS05, Thm. 1.8]. By Theorem 5.2 again, $\mathbb{G}' \leq \mathbb{G}$ is polynomial if and only if it is conjugate into some subgroup in $\mathcal{P}(\mathbb{G})$. Uniqueness of $\mathcal{P}(\mathbb{G})$ follows from malnormality of peripheral structures. □

Not only is $\mathcal{P}(\mathbb{G})$ a group invariant of $\mathbb{G}$, it turns out to be a geometric invariant! Behrstock–Druţu–Mosher [BDM09, Thm. 4.8] strengthened Druţu's geometric invariance theorem when the group is hyperbolic relative to non-(relatively hyperbolic) subgroups:

Theorem 5.3. *If $f\colon \mathrm{Cay}(\mathbb{G}) \to \mathrm{Cay}(\mathbb{G}')$ is a quasi-isometry and $\mathbb{P} \in \mathcal{P}(\mathbb{G})$, then $f(\mathbb{P})$ is a finite Hausdorff distance from a conjugate of some $\mathbb{P}' \in \mathcal{P}(\mathbb{G}')$.* □

To conclude, we return to the extreme case when ϕ is atoroidal. The outer automorphism ϕ is fully irreducible if there are no ϕ-periodic conjugacy classes of nontrivial proper free factors of $\mathbb{F}$. Using Lemma 3.5, we [Mut21, Thm 4.3] proved that being both fully irreducible and atoroidal was a group (actually commensurability) invariant:

Theorem 5.4. *$\mathbb{G}$ has no infinite index free-by-cyclic subgroups if and only if ϕ is fully irreducible and atoroidal.* □

In [Mut21, p. 48], we conjecture that being fully irreducible and atoroidal is a geometric invariant, but it is not clear what the equivalent property of the Cayley graph would be. In fact, we suspect something more general! Recall the definition of *height* in Section 1: the length of the longest properly nested sequences of attracting laminations for ϕ.

Conjecture 5.5. *If $\mathbb{G}, \mathbb{G}'$ are quasi-isometric, then $\mathfrak{h}(\phi) = \mathfrak{h}(\psi)$.*

The height 0 case of the conjecture is precisely Corollary 4.9. For atoroidal outer automorphisms, being fully irreducible may be equivalent to having height 1 and no fixed free splittings; thus, the height 1 case of the conjecture may be equivalent to the geometric invariance of being fully irreducible and atoroidal.

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