Finding Relative Immersions on Free Groups

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Topological representatives on free groups

Given an injective endomorphism $\phi : F_n \to F_n$, what is the best way to represent ϕ as a topological map?

Needed:

- a graph X,
- an isomorphism $\alpha : F_n \to \pi_1(X)$ (**marking**),
- a map $f : X \to X$ that induces $[\phi]$:

$$\begin{array}{ccc} F_n & \xrightarrow{\ \phi\ } & F_n \\ \alpha\downarrow & & \downarrow\alpha \\ \pi_1(X) & \xrightarrow[f_*]{} & \pi_1(X) \end{array}$$

commutes up to post-composition with an inner automorphism

- can read $[\phi]$'s dynamics of f.

Automorphisms of free groups

$f : X \to X$ is a homotopy equivalence.

Is $f \simeq$ a homeomorphism?

$f \simeq$ graph symmetry if and only if $[\phi]$ has finite order.
(Culler, Khramtsov, Zimmermann)

Automorphisms of free groups

$f : X \to X$ is a homotopy equivalence and assume $[\phi]$ has infinite order.

Is $f \simeq$ a locally injective map (**immersion**)?

No: immersion $f \implies \tilde{f} : \tilde{X} \to \tilde{X}$ injective

surjective $\phi \implies \tilde{f}$ simplicial homeomorphism

$\implies f$ is a homeomorphism.

Automorphisms of free groups

$f : X \to X$ is a homotopy equivalence and assume $[\phi]$ has infinite order. What's the best we can hope for?

(Bestvina-Handel) $f \simeq$ a relative train track

$\longrightarrow$ Scott conjecture

(Bestvina-Feighn-Handel) $f \simeq$ improved rel. train track

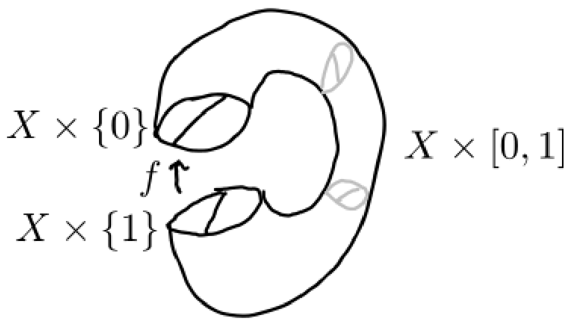
$\longrightarrow$ Tits alternative for $\mathrm{Out}(F_n)$

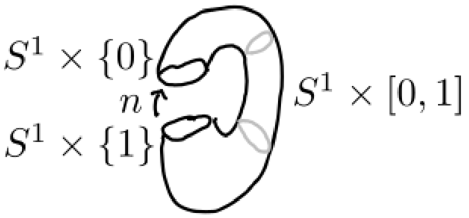
Automorphisms of free groups

$f : X \to X$ is a homotopy equivalence and assume $[\phi]$ has infinite order. What's the best we can hope for?

(Brinkmann) The following are equivalent:

1. $[\phi]$ is *atoroidal*;
2. $F_n \rtimes_\phi \mathbb{Z}$ is *word-hyperbolic*;
3. $F_n \rtimes_\phi \mathbb{Z}$ has no $BS(1, n)$ subgroups;
4. $F_n \rtimes_\phi \mathbb{Z}$ has no $\mathbb{Z}^2$ subgroups.





Nonsurjective endomorphisms

$f : X \to X$ is π_1-injective.

Is $f \simeq$ an improved relative train track? No!

Is $f \simeq$ a relative train track? Yes, but not enough!

Is $f \simeq$ an immersion? Most times!

(Reynolds) If ϕ is nonsurjective and irreducible, then $f \simeq$ a unique expanding local homothety.

Examples and non-examples

Examples: $n = 1, \quad \phi : a \mapsto a^2$

$X = S^1$

$f =$ degree 2 map

$n = 2, \quad \phi : \begin{cases} a \mapsto ab \\ b \mapsto ba \end{cases}$

$X =$ 

$, \alpha : \begin{cases} a \mapsto [l] \\ b \mapsto [r] \end{cases}$

$f =$ obvious map

Non-examples: $n = 3, \quad \phi : \begin{cases} a \mapsto b \\ b \mapsto ab \\ c \mapsto c^2 \end{cases}$

$\phi|_{\langle a,b \rangle} : \langle a, b \rangle \to \langle a, b \rangle$ is an infinite order automorphism $\implies f \not\simeq$ an immersion.

$n = 2, \quad \phi : \begin{cases} a \mapsto a \\ b \mapsto ab^2 \end{cases}$

$\phi|_{\langle a \rangle} : \langle a \rangle \to \langle a \rangle$ is an automorphism $\implies f \not\simeq$ an expanding immersion.

Other reasons $\implies f \not\simeq$ an immersion.

My results

Theorem 1. (Mutanguha)

If ϕ is nonsurjective, then $f \simeq$ a relative expanding immersion.

$$n = 3, \quad \phi : \begin{cases} a \mapsto b \\ b \mapsto ab \\ c \mapsto c^2 \end{cases}$$

$X : \langle a, b \rangle$

$f : \langle b, ba \rangle$

$$n = 2, \quad \phi : \begin{cases} a \mapsto a \\ b \mapsto ab^2 \end{cases}$$

$X : \langle a \rangle$

$f : \langle a \rangle$

Corollary. (M.) *$f \simeq$ expanding immersion $\iff$ $[\phi]$ has no fixed proper free factor system.*

Theorem 2. (M.)

$F_n *_\phi$ is word-hyperbolic $\iff$ it has no $BS(1, n)$ subgroups.

Outer space and its spine

$CV_n = \{(X,\alpha) \text{ marked metric graphs }\}/\sim$

$(X_1,\alpha_1) \sim (X_2,\alpha_2)$ if

$$\begin{array}{ccc} & & \pi_1(X_1) \\ & \overset{\alpha_1}{\nearrow} & \\ F_n & & \Big\downarrow g_* \text{ homothety} \\ & \underset{\alpha_2}{\searrow} & \\ & & \pi_1(X_2) \end{array}$$

with $[g_* \circ \alpha_1] = [\alpha_2]$.

Spine of $CV_n = CV_n/\sim'$ where we forget the metrics

$=$ vertices of a locally finite simplicial complex

The action on the spine

$[\phi]$ an outer endomorphism

$[X, \alpha]$ a marked graph in the spine

$$\begin{array}{ccc} F_n & \xrightarrow{\alpha'} & \pi_1(X') \\ \phi \downarrow & & \downarrow i_* \text{ immersion} \\ F_n & \xrightarrow[\alpha]{} & \pi_1(X) \end{array}$$

Set $[X, \alpha] \cdot [\phi] = [X', \alpha']$

Lemma.

Fixed point in the spine $\iff$ $f \simeq$ *immersion.*

Fixed point in outer space $\iff$ $f \simeq$ *local homothety.*

Sketch proof of Reynolds' theorem

(Reynolds) If ϕ is nonsurjective and irreducible, then its action on outer space has a unique fixed point.

1) Choose a marked graph $[X, \alpha]$ and iterate $[X, \alpha] \cdot [\phi]^n$

2) The sequence is eventually periodic (in the spine)

3) There's a fixed point next to the periodic points

4) ...

$$n = 2, \quad \phi : \begin{cases} a \mapsto ab \\ b \mapsto ba \end{cases}$$

$X =$ 

$$, \alpha : \begin{cases} a \mapsto [l] \\ b \mapsto [r] \end{cases}$$

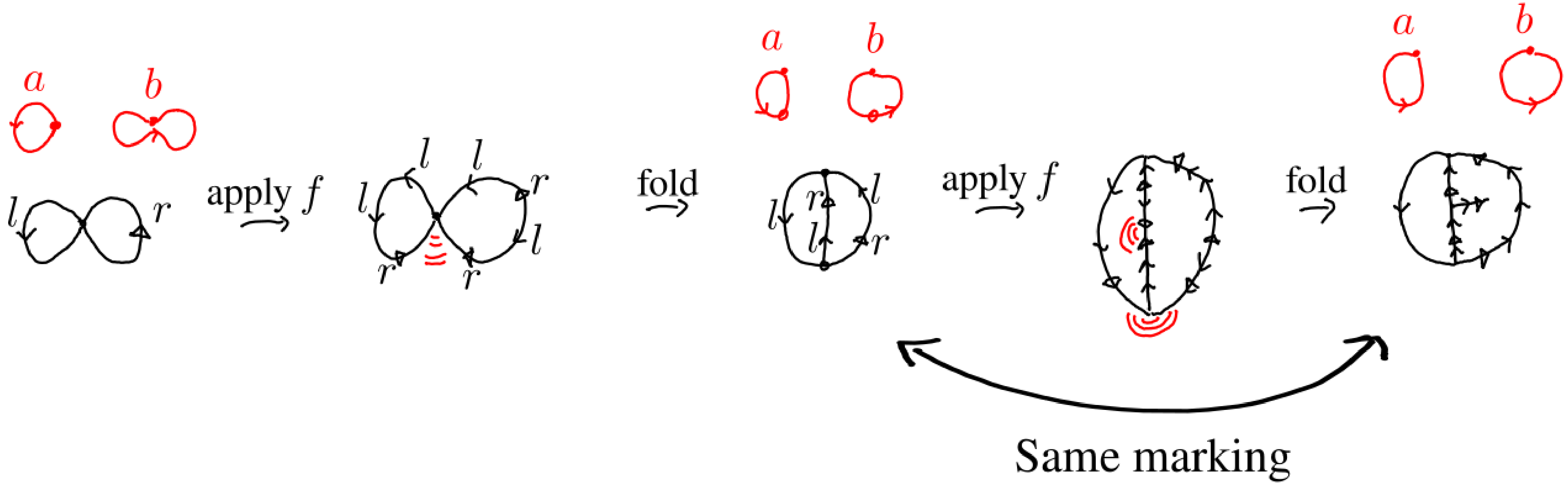
$f =$ obvious map is an immersion

$[X, \alpha]$ is a fixed point in the spine

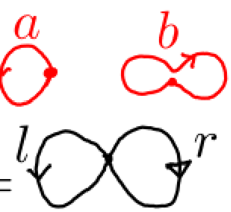
Sketch proof of Reynolds' theorem

$$n = 2, \quad \phi : \begin{cases} a \mapsto ab \\ b \mapsto ba \end{cases}, \qquad X =$$ ,

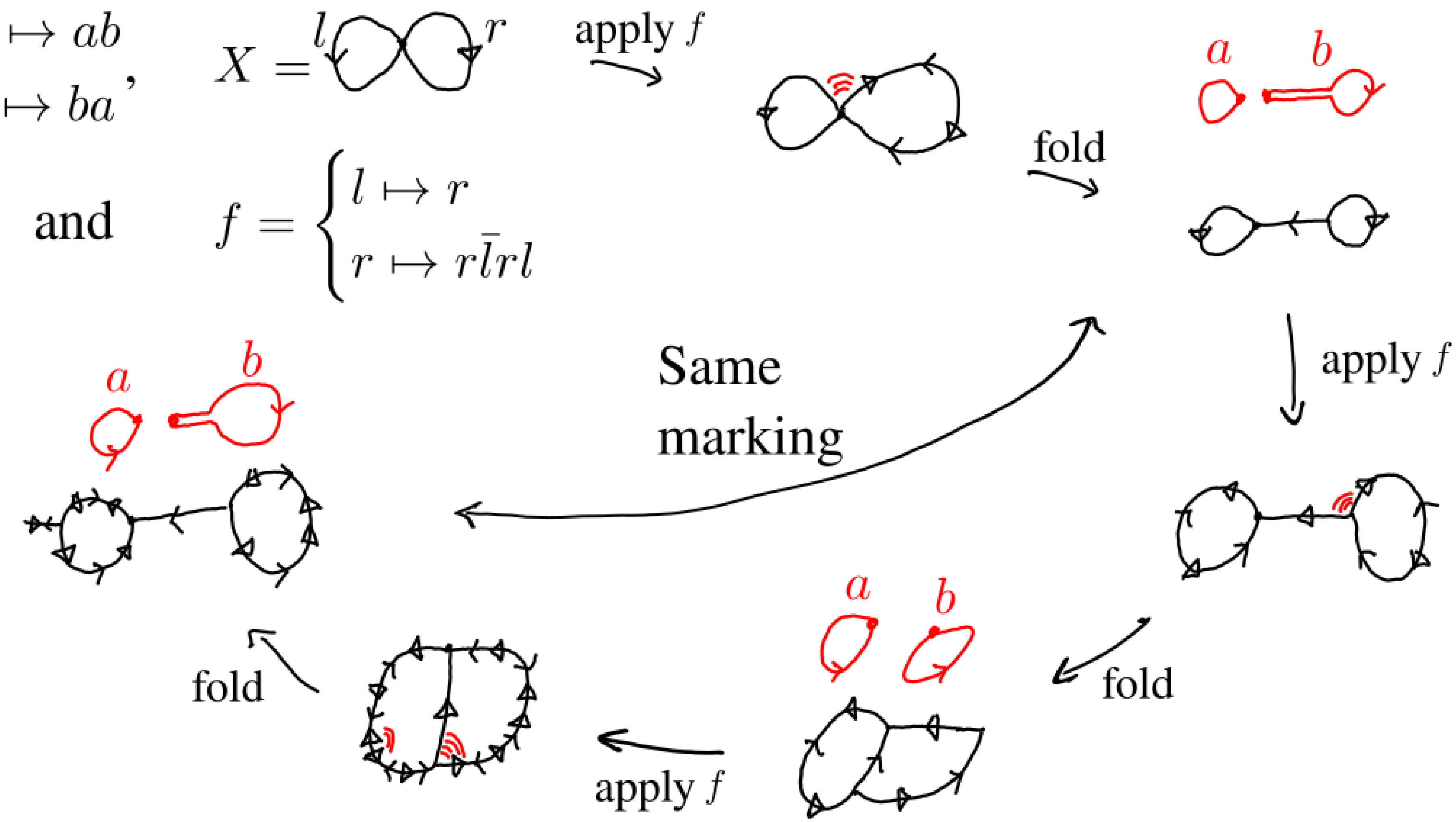
$$\alpha : \begin{cases} a \mapsto [l] \\ b \mapsto [lr] \end{cases} \quad \text{and} \quad f = \begin{cases} l \mapsto llr \\ r \mapsto \bar{r}\bar{l}rl \end{cases}$$



Sketch proof of Reynolds' theorem

$$n = 2, \quad \phi : \begin{cases} a \mapsto ab \\ b \mapsto ba \end{cases}, \qquad X = $$ 

$$\alpha : \begin{cases} a \mapsto [l] \\ b \mapsto [\bar{l}r] \end{cases} \quad \text{and} \quad f = \begin{cases} l \mapsto r \\ r \mapsto r\bar{l}rl \end{cases}$$



How to use prove things for injective endomorphisms

Step 1. Prove the result for automorphisms.

Step 2. Prove the result for expanding immersions.

Step 3. "Relativize" Step 2 to combine with Step 1 using relative expanding immersions.

Step 4. Celebrate!

Thank you!