Hyperbolic hyperbolic-by-cyclic groups are cubulable

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Abstract

We show that the mapping torus of a hyperbolic group by a hyperbolic automorphism is cubulable. Along the way, we (i) give an alternate proof of Hagen and Wise's theorem that hyperbolic free-by-cyclic groups are cubulable, and (ii) extend to the case with torsion Brinkmann's thesis that a torsion-free hyperbolic-by-cyclic group is hyperbolic if and only if it does not contain $\mathbb{Z}^2$-subgroups.

1 Introduction

In this note, we prove the following:

Theorem (Corollary 5.4)**.** *Hyperbolic hyperbolic-by-cyclic groups are cubulable.*

A *hyperbolic-by-cyclic* group is a semidirect product $G \rtimes \mathbb{Z}$ of a hyperbolic group G with the integers $\mathbb{Z}$. A group is *cubulable* if it admits an isometric action on a CAT(0) cube complex that is cubical, proper, and cocompact. The repetition in the statement is intended: we assume that both G and $G \rtimes \mathbb{Z}$ are hyperbolic (equivalently, G is hyperbolic and $G \rtimes \mathbb{Z}$ does not contain $\mathbb{Z}^2$, see Corollary 5.3). This restricts what G can be.

Emblematic cases of our theorem are known by outstanding works. First and foremost, if G is a closed surface group, then any hyperbolic extension $G \rtimes \mathbb{Z}$ is a closed hyperbolic 3-manifold group [Thu82]. Its cubulation is due to independent works of Bergeron and Wise [BW12]

— using Kahn and Markovic's surface subgroup theorem [KM12], and Dufour [Duf12] — using the immersed quasiconvex surfaces of Cooper, Long, and Reid [CLR94]. Second, when G is free, Hagen and Wise cubulated the mapping torus $G \rtimes \mathbb{Z}$ of a fully irreducible hyperbolic automorphism [HW16].

Hagen and Wise also treat extensions of free groups by arbitrary hyperbolic automorphisms in [HW15a], a notoriously difficult analysis. We do not rely on, nor follow, that work. Instead, our proof uses the emblematic cases above in a telescopic argument that encompasses the case when G is a torsion-free hyperbolic group (see Theorem 4.2). It provides a hopefully appreciated alternative.

We adopt a relative viewpoint and bootstrap the relative cubulation of certain free-product-by-cyclic groups by the first two named authors [DM]; this uses recent work of Groves and Manning on improper actions on CAT(0) cube complexes [GM] along with the malnormal combination theorem of Hsu and Wise [HW15b]. The need for the theory of train tracks (of free groups or free product automorphisms, see [BH92, FM15]) is limited to absolute train tracks for the fully irreducible case; it is encapsulated in the relative cubulation of free-product-by-cyclic groups [DM].

For a hyperbolic group G possibly with torsion, if there exists a hyperbolic extension $G \rtimes \mathbb{Z}$, then G is virtually torsion-free (and residually finite) by Proposition 5.2. In particular, $G \rtimes \mathbb{Z}$ is virtually cubulable hyperbolic, and hence cubulable [Wis21, Lem. 7.14]. As a consequence, we have:

Corollary. *If a hyperbolic-by-cyclic group Γ is hyperbolic, then:*

1. *Γ is virtually (compact) special [Ago13];*
2. *Γ is $\mathbb{Z}$-linear and its quasiconvex subgroups are separable [HW08];*
3. *Γ virtually surjects onto F_2 [AM15];*
4. *Γ is conjugacy separable [MZ16]; and*
5. *Γ admits Anosov representations [DFWZ23].*

We end this introduction with a question. Proposition 5.2 states that a hyperbolic group is virtually a free product of free and surface groups whenever it admits a hyperbolic automorphism. However, the converse is false as can be seen from a hyperbolic triangle group or the free product of two finite groups — these have finite outer automorphism groups.

Question. *Is there an algebraic characterisation of hyperbolic groups that admit hyperbolic automorphisms?*

Note that Pettet characterised virtually free groups with finite outer automorphism groups [Pet97].

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2 Free factor systems

A *free decomposition* of a group G is an isomorphism $G \cong A_1 * \cdots * A_k * F_r$, where $k \geqslant 0$, $r \geqslant 0$, each *peripheral free factor* A_i is not trivial, and F_r is free with rank r. We call $\mathcal{A} = (A_1, \ldots, A_k)$ a *free factor system* of G; it is *proper* unless $k \leqslant 1$ and $r = 0$. The integer $k + r$ is the *Kurosh co-rank* of the free factor system $\mathcal{A}$. A nontrivial group is *freely indecomposable* if its free factor systems have Kurosh co-rank 1.

Assume G is finitely generated for the rest of this section. A *Grushko decomposition* of G is a free decomposition whose free factor system $\mathcal{A}$ has maximal Kurosh co-rank and peripheral free factors A_i are not $\mathbb{Z}$; in that case, we call $\mathcal{A}$ a *Grushko free factor system* and its Kurosh co-rank is the *Kurosh–Grushko rank* of G.

Recall the preorder of free factor systems of G: a free factor system $\mathcal{B} = (B_1, \ldots, B_\ell)$ is lower than $\mathcal{A}$ if each B_j is conjugate in G to a subgroup of some A_i. In this case, a free decomposition with peripherals $\mathcal{A}$ refines to one with peripherals $\mathcal{B}$ (as seen by the actions of A_i on $T_{\mathcal{B}}$, a Serre tree whose nontrivial vertex stabilisers are exactly the conjugates of all B_j), and the Kurosh co-rank of $\mathcal{B}$ is at least that of $\mathcal{A}$ (see [DL22, Lem. 1.1] for a similar argument); if it is equal, then $\mathcal{A}$ is also lower than $\mathcal{B}$.

Let $\mathcal{B} = (B_1, \ldots, B_\ell)$ be a free factor system of G. A *proper* $(G, \mathcal{B})$*-free factor* is a nontrivial point stabiliser of a nontrivial action of G on a tree, for which edge stabilisers are trivial, and in which each B_j is elliptic. In other words, it is a peripheral free factor A_i in a free factor system $\mathcal{A}$ that is higher than $\mathcal{B}$ in the preorder.

A minimal free factor system in this preorder is a Grushko free factor system; it is unique up to the preorder's equivalence relation. So any automorphism preserves the Grushko free factor system $(A_1, \ldots, A_k)$, i.e. it sends each A_i to a conjugate of some A_j. A free factor system is *periodic* with respect to $\phi \in \mathrm{Aut}(G)$ if some (positive) power of ϕ preserves it.

Lemma 2.1. *Suppose G is a finitely generated group. If $\mathcal{B} = (B_1, \ldots, B_\ell)$ is a proper free factor system, then each B_i has Kurosh–Grushko rank strictly lower than the Kurosh–Grushko rank of G.*

If G has Kurosh–Grushko rank $\geqslant 2$, then any automorphism $\phi\colon G \to G$ has a free factor system that is maximal among ϕ-periodic proper free factor systems.

Proof. Since $\mathcal{B}$ is proper, $G \cong B_i * H$ for some nontrivial group H. By uniqueness of the Grushko decomposition, the Kurosh–Grushko rank of G is the sum of those of B_i and H.

For the second assertion, as the Kurosh–Grushko rank is at least 2, the Grushko free factor system is proper and ϕ-periodic. Restricting to ϕ-periodic proper free factor systems, any one with the lowest Kurosh co-rank is maximal in the preorder. □

3 Ingredients

Let G be a torsion-free group. For this section, we assume:

- a free factor system $\mathcal{B} = (B_1, \ldots, B_\ell)$ has Kurosh co-rank $\geqslant 3$;
- an automorphism $\psi\colon G \to G$ preserves $\mathcal{B}$, denoted $\psi \in \mathrm{Aut}(G, \mathcal{B})$;
- $\psi \in \mathrm{Aut}(G, \mathcal{B})$ is *relatively fully irreducible*, i.e. any ψ-periodic (up to conjugacy) proper $(G, \mathcal{B})$-free factor must be conjugate to some B_i;
- $\psi \in \mathrm{Aut}(G, \mathcal{B})$ is *relatively atoroidal*, i.e. any ψ-periodic conjugacy class of nontrivial elements in G intersects some B_i.

Here is an equivalent definition of relatively fully irreducible:

Lemma 3.1. *An automorphim $\psi \in \mathrm{Aut}(G, \mathcal{B})$ is relatively fully irreducible if and only if $\mathcal{B}$ is a maximal ψ-periodic proper free factor system.*

Proof. If some ψ-periodic proper free factor system $(A_1, \ldots, A_k)$ is strictly higher than $\mathcal{B} = (B_1, \ldots, B_\ell)$ in the preorder, then some A_i is a ψ-periodic proper $(G, \mathcal{B})$-free factor that is not conjugate to any B_j.

Conversely, if some ψ-periodic proper $(G, \mathcal{B})$-free factor A_1 is not conjugate to any B_i, then the ψ-periodic free factor system (A_1) can be extended to a ψ-periodic proper free factor system $(A_1, \ldots, A_k)$ that is strictly higher than $\mathcal{B}$ by including some (conjugates of) B_i. □

For $h \in G$, $\mathrm{ad}_h \colon G \to G$ denotes the inner automorphism $g \mapsto hgh^{-1}$. For a peripheral free factor B_i, let $k_i \geqslant 1$ be the smallest integer such that $\psi^{k_i}(B_i) = g_i^{-1} B_i g_i$ for some $g_i \in G$. The *peripheral suspension* $B_i \rtimes \mathbb{Z}$ is the suspension of B_i by $\mathrm{ad}_{g_i} \circ \psi^{k_i}|_{B_i} \colon B_i \to B_i$; this group naturally embeds in $G \rtimes_\psi \mathbb{Z}$ — one can verify using normal forms that the natural homomorphism $B_i \rtimes \langle s \rangle \to G \rtimes_\psi \langle t \rangle$ that maps $s \mapsto g_i t^{k_i}$ is injective.

The first two named authors recently gave a *relative cubulation* (introduced in [EG20]) of the mapping torus of a relatively fully irreducible relatively atoroidal automorphism. Their proof is adapted from Hagen and Wise's cubulation of hyperbolic irreducible free-by-cyclic groups [HW16].

Theorem 3.2 (cf. [DM, Thm. 1.1])**.** *Under this section's assumptions, the mapping torus $G \rtimes_\psi \mathbb{Z}$ acts cocompactly on a CAT(0) cube complex, where each cell stabiliser is either trivial or conjugate to a finite index subgroup of some peripheral suspension $B_i \rtimes \mathbb{Z}$.* □

The cited theorem has an additional assumption, absence of twinned subgroups: two subgroups $H_1 \neq H_2$ of G are *twinned* in $\mathcal{B}$ if they are conjugates of some B_j, B_k and $\mathrm{ad}_g \circ \psi^n(H_i) = H_i$ $(i = 1, 2)$ for some $n \geqslant 1$ and $g \in G$. This assumption ensures the family of peripheral suspensions is malnormal (for relative hyperbolicity [DL22, Thm. 0.1]), but Guirardel remarked that it is redundant:

Lemma 3.3 (Guirardel)**.** *As $\mathcal{B}$ has Kurosh co-rank $\geqslant 3$ and $\psi \in \mathrm{Aut}(G, \mathcal{B})$ is relatively fully irreducible, there are no twinned subgroups in $\mathcal{B}$.*

Our proof of the lemma uses objects (expanding train tracks, limit trees, geometric trees of surface type) that we do not define here for the sake of brevity; we refer the reader to the cited literature for each.

Proof. The automorphism ψ is represented by an expanding irreducible train track (see [DL22, Sec. 1.3]). Projectively iterating the train track produces the limit $(G, \mathcal{B})$-tree T and a ψ-equivariant expanding homothety $h \colon T \to T$ (see [BFH97, p. 232]). Note that nontrivial point stabilisers of T are ψ-periodic (up to conjugacy) by the finiteness of G-orbits of branch points in T [Hor17, Cor. 5.5] and the ψ-equivariance of h.

Let $H \leqslant G$ be a nontrivial nonperipheral point stabiliser of T — nonperipheral means the subgroup is not conjugate to some B_i. Then no proper $(G, \mathcal{B})$-free factor contains H — otherwise, the smallest such factor would be nonperipheral and ψ-periodic, yet $\psi \in \mathrm{Aut}\,(G, \mathcal{B})$ is relatively fully irreducible. Thus T is geometric of surface type [Hor17, Sec. 6.2, Lem. 6.8] and the point stabiliser H is cyclic [Hor17, Prop. 6.10]. As H was arbitrary, all nonperipheral point stabilisers of T are cyclic; therefore, there are no twinned subgroups in $\mathcal{B}$ because they would generate a noncyclic nonperipheral T-elliptic subgroup by the ψ-equivariance of h. □

We will use the following theorem of Groves and Manning to upgrade relative cubulations in the next section.

Theorem 3.4 (cf. [GM, Thm. D])**.** *If a hyperbolic group Γ acts cocompactly on a CAT(0) cube complex so that cell stabilisers are quasiconvex and cubulable, then Γ is cubulable.* □

The cited theorem has "virtually special" in place of "cubulable". Since virtually cubulable hyperbolic groups are cubulable [Wis21, Lem. 7.14], the properties "virtually special" and "cubulable" are equivalent for hyperbolic groups by Agol's theorem [Ago13]. In particular, for hyperbolic groups, being cubulable is a commensurability invariant.

Finally, for sporadic cases when the Kurosh co-rank is 2, we will need a specialisation of Hsu and Wise's malnormal combination theorem:

Theorem 3.5 (cf. [HW15b, Cor. C])**.** *Suppose $\Gamma = \Gamma_1 *_{\langle c \rangle} \Gamma_2$ or $\Gamma_1 *_{\langle c \rangle}$ is hyperbolic and $\langle c \rangle$ is an infinite cyclic malnormal subgroup of Γ. If each Γ_i is cubulable, then Γ is cubulable.* □

The two decompositions can be stated together as: "Γ splits over $\langle c \rangle$."

4 The bootstrap

The following proposition is due to Sela (see Proposition 5.1 for a proof).

Proposition 4.1 (cf. [Sel97, Cor. 1.10]). *Assume G is a torsion-free hyperbolic group and some extension $G \rtimes_\phi \mathbb{Z}$ does not contain a copy of $\mathbb{Z}^2$. If G is freely indecomposable, then it is the fundamental group of a closed surface.* □

We may now prove the central result of this note:

Theorem 4.2. *Let G be a torsion-free hyperbolic group. If $G \rtimes_\phi \mathbb{Z}$ is hyperbolic, then it is cubulable.*

Proof. We proceed by induction on the Kurosh–Grushko rank.

If the Kurosh–Grushko rank of G is 1, then G is freely indecomposable. By Proposition 4.1, G is a closed surface group and, by the classification of its automorphisms, ϕ is pseudo-Anosov [Thu82, Thm. 5.5]. Then $G \rtimes_\phi \mathbb{Z}$ is famously the fundamental group of a closed hyperbolic 3-manifold [Thu82, Thm. 5.6] and cubulable, as already mentioned in Section 1. Assume $n \geqslant 2$ and the theorem holds for torsion-free hyperbolic groups of Kurosh–Grushko rank $< n$.

Let the Kurosh–Grushko rank of G be n. Lemma 2.1 provides a maximal ϕ-periodic proper free factor system $\mathcal{B} = (B_1, \ldots, B_\ell)$, and each B_i has Kurosh–Grushko rank $< n$. As each peripheral free factor B_i is quasiconvex in the hyperbolic group G, a closest point projection $G \to B_i$ is Lipschitz and extends (cosetwise) to a *peripheral retraction* $G \rtimes_\phi \mathbb{Z} \to B_i \rtimes \mathbb{Z}$ to the peripheral suspension. Since ϕ is a quasi-isometry, the peripheral retractions are Lipschitz by the Morse lemma (in G) — a variation of this idea appears in [Mit98, Sec. 3]. Thus the peripheral suspensions are quasiconvex and hyperbolic. By the induction hypothesis, each $B_i \rtimes \mathbb{Z}$ is cubulable.

We distinguish two cases. The first case is when the Kurosh co-rank of $\mathcal{B}$ is at least 3. Some positive power ψ of ϕ preserves $\mathcal{B}$ and, by Lemma 3.1, $\psi \in \mathrm{Aut}\,(G, \mathcal{B})$ is relatively fully irreducible. Since $G \rtimes_\psi \mathbb{Z}$ is hyperbolic, it has no $\mathbb{Z}^2$-subgroups and there are no ψ-periodic conjugacy classes of nontrivial elements in G. In particular, $\psi \in \mathrm{Aut}\,(G, \mathcal{B})$ is relatively atoroidal. By Theorem 3.2, $G \rtimes_\psi \mathbb{Z}$ acts cocompactly on a CAT(0) cube complex, where each cell stabiliser is either trivial or conjugate to a finite index subgroup of some quasiconvex cubulable $B_i \rtimes \mathbb{Z}$. Groves and Manning's Theorem 3.4 thus implies $G \rtimes_\psi \mathbb{Z}$ is cubulable. It naturally embeds in $G \rtimes_\phi \mathbb{Z}$ with finite index, so the latter is also cubulable by [Wis21, Lem. 7.14].

The last case is when the Kurosh co-rank of $\mathcal{B}$ is 2. There are three possibilities: G is $B_1 * B_2$, $B_1 * F_1$, or F_2. We rule out the third possibility as $F_2 \rtimes \mathbb{Z}$ is never hyperbolic — it is a classical theorem of Nielsen that any automorphism of F_2 maps the commutator of a basis to a conjugate of itself or its inverse [Nie17]. To conclude, we will prove that $\Gamma = G \rtimes_\phi \langle t \rangle$ (virtually) satisfies the hypotheses of Hsu and Wise's Theorem 3.5, and hence is cubulable. Note that $\langle t \rangle$ is a maximal cyclic subgroup of Γ, and hence malnormal. It remains to show that Γ splits over $\langle t \rangle$ as needed.

In the first possibility, up to taking the square of ϕ, we may assume that ϕ preserves the conjugacy classes of both B_1 and B_2. After conjugation (which does not change the mapping torus), we may assume it fixes B_1 (setwise) and, being an automorphism, it sends B_2 to a conjugate by an element of B_1. After further conjugation, it fixes both B_1 and B_2. Then the mapping torus $\Gamma = (B_1 * B_2) \rtimes_\phi \langle t \rangle \cong (B_1 \rtimes \langle t \rangle) *_{\langle t \rangle} (B_2 \rtimes \langle t \rangle)$.

In the second possibility, we write $G = B_1 * \langle s \rangle$. Up to taking the square of ϕ and composing with a conjugation, we may assume that $\phi(B_1) = B_1$ and $\phi(s) = sb$ for some $b \in B_1$. Consider $G \rtimes_\phi \langle t \rangle$, where one has the relation $tst^{-1} = sb$, or written differently $s^{-1}ts = bt$. Then, rewriting the presentation, one has that

$$\Gamma = (B_1 * \langle s \rangle) \rtimes_\phi \langle t \rangle \cong (B_1 \rtimes \langle t \rangle)*_{\langle t \rangle^s = \langle bt \rangle},$$

where the last operation is an HNN extension with a stable letter s that (right) conjugates $\langle t \rangle$ to $\langle bt \rangle$ (and actually t to bt). □

5 Once more, with torsion

Now G is a finitely presented group (possibly with torsion). It has a maximal decomposition as the fundamental group of a finite graph of groups with finite edge groups [Dun85]. The infinite vertex groups are thus one-ended [Sta71]. We call this a *Dunwoody–Stallings decomposition*. It is not unique, but the conjugacy classes of infinite vertex groups are uniquely defined: they are conjugacy classes of the maximal one-ended subgroups of G. The following is a generalisation of Proposition 4.1:

Proposition 5.1. *Assume G is a hyperbolic group (possibly with torsion) and some extension $G \rtimes_\phi \mathbb{Z}$ does not contain a copy of $\mathbb{Z}^2$. Then every maximal one-ended subgroup of G is virtually a closed surface group.*

Proof. Let H be a maximal one-ended subgroup of G. Since there are only finitely many conjugacy classes of such subgroups, $\psi = (\mathrm{ad}_g \circ \phi^k)|_H$ is an automorphism of H for some integer $k \geqslant 1$ and element $g \in G$.

Similar to the discussion in Section 3, the suspension $H \rtimes_\psi \mathbb{Z}$ naturally embeds in $G \rtimes_\phi \mathbb{Z}$. As H is one-ended, its JSJ decomposition is preserved by ψ [Bow98, Thm. 0.1]. The lack of $\mathbb{Z}^2$ in $G \rtimes_\phi \mathbb{Z}$ imposes that the JSJ is trivial but not a rigid vertex [BF95, Cor. 1.3]. It is therefore a vertex of surface type. In particular, H is virtually a closed surface group (see, for instance, [Mar07, Sec. 4]). □

We are now ready to state the main observation of this section.

Proposition 5.2. *If G is a hyperbolic group (possibly with torsion) and some extension $G \rtimes_\phi \mathbb{Z}$ does not contain a copy of $\mathbb{Z}^2$, then G has a characteristic finite index subgroup that is a free product of closed surface groups and free groups. In particular, G is residually finite.*

Proof. Let $\mathbb{X}$ be a Dunwoody–Stallings decomposition of G. We need notations for the decomposition: the underlying finite graph is X; for each vertex v in X, its vertex group is $\mathbb{X}_v$; and for each edge e in X, its finite edge group is $\mathbb{X}_e$. For each vertex v, denote by H_v a normal finite index subgroup of $\mathbb{X}_v$ that is either trivial or a closed surface group, as guaranteed by Proposition 5.1.

As the subgroups H_v are torsion-free, the surjections $q_v \colon \mathbb{X}_v \to \mathbb{X}_v / H_v$ are injective on finite subgroups. Thus we define a graph of finite groups $\mathbb{Y}$ with underlying graph X, vertex groups $\mathbb{X}_v / H_v$, and edge groups $\mathbb{X}_e$; the surjections q_v induce a surjection $q \colon G \to \pi_1(\mathbb{Y})$ with a torsion-free kernel. The quotient $\pi_1(\mathbb{Y})$ is virtually free by Karrass, Pietrowski, and Solitar's characterisation [KPS73, Thm. 1].

Let $J \leqslant \pi_1(\mathbb{Y})$ be a free finite index subgroup. Since J and the kernel of q are torsion-free, the preimage $q^{-1}(J) \leqslant G$ is a torsion-free finite index subgroup. The intersection H of subgroups of G with index $[G : q^{-1}(J)]$ is a characteristic torsion-free finite index subgroup. The decomposition $\mathbb{X}$ of G induces a Grushko decomposition of H whose freely indecomposable free factors are closed surface groups. □

We may extend Brinkmann's thesis [Bri00] to the case with torsion.

Corollary 5.3. *Suppose G is a hyperbolic group. Then $G \rtimes_\phi \mathbb{Z}$ is hyperbolic if and only if it does not contain a copy of $\mathbb{Z}^2$.*

The forward implication is standard. Conversely, if $G \rtimes_\phi \mathbb{Z}$ does not contain a copy of $\mathbb{Z}^2$, then the same holds for the finite index subgroup

$G_0 \rtimes_{\phi|_{G_0}} \mathbb{Z}$, where G_0 is the torsion-free subgroup given by Proposition 5.2. As $G_0 \rtimes_{\phi|_{G_0}} \mathbb{Z}$ is hyperbolic [Bri00], so is $G \rtimes_\phi \mathbb{Z}$.

Corollary 5.4. *If G and $G \rtimes_\phi \mathbb{Z}$ are hyperbolic groups, then $G \rtimes_\phi \mathbb{Z}$ is cubulable.*

Again, consider the finite index subgroup $G_0 \rtimes_{\phi|_{G_0}} \mathbb{Z}$ of $G \rtimes_\phi \mathbb{Z}$, where G_0 is given by Proposition 5.2. $G_0 \rtimes_{\phi|_{G_0}} \mathbb{Z}$ is cubulable by Theorem 4.2, and hence, by [Wis21, Lem. 7.14], so is $G \rtimes_\phi \mathbb{Z}$.

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