Hyperbolic hyperbolic-by-cyclic groups are cubulable

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Abstract

We show that the mapping torus of a hyperbolic group by a hyperbolic automorphism is cubulable. Along the way, we (i) give an alternate proof of Hagen and Wise's theorem that hyperbolic free-by-cyclic groups are cubulable, and (ii) extend to the case with torsion Brinkmann's thesis that a torsion-free hyperbolic-by-cyclic group is hyperbolic if and only if it does not contain $\mathbb{Z}^2$-subgroups.

1 Introduction

In this note, we prove the following:

Theorem. *All hyperbolic hyperbolic-by-cyclic groups are cubulable.*

A *hyperbolic-by-cyclic* group is a semidirect product $G \rtimes \mathbb{Z}$ of a hyperbolic group G with the integers $\mathbb{Z}$. A group is *cubulable* if it admits an isometric action on a CAT(0) cube complex that is cubical, proper, and cocompact. The repetition in the statement is intended: we assume that both G and $G \rtimes \mathbb{Z}$ are hyperbolic (equivalently, G is hyperbolic and $G \rtimes \mathbb{Z}$ does not contain $\mathbb{Z}^2$, see Corollary 5.3). This restricts what G can be.

Emblematic cases of our theorem are known by outstanding works. First and foremost, if G is a closed surface group, then any hyperbolic extension $G \rtimes \mathbb{Z}$ is a closed hyperbolic 3-manifold group [Thu82]. Its cubulation is due to independent works of Bergeron and Wise [BW12]

— using Kahn and Markovic's surface subgroup theorem [KM12], and Dufour [Duf12] — using the immersed quasiconvex surfaces of Cooper, Long, and Reid [CLR94]. Second, when G is free, Hagen and Wise cubulated the mapping torus $G \rtimes \mathbb{Z}$ of a fully irreducible hyperbolic automorphism [HW16].

Hagen and Wise also treat extensions of free groups by arbitrary hyperbolic automorphisms in [HW15a], a notoriously difficult analysis. We do not rely on, nor follow, that work. Instead, our proof uses the emblematic cases above in a telescopic argument that encompasses the case when G is a torsion-free hyperbolic group (see Theorem 4.2). It provides a hopefully appreciated alternative.

We adopt a relative viewpoint and bootstrap the relative cubulation of certain free-product-by-cyclic groups by the first two named authors [DM22]; this uses recent work of Groves and Manning on improper actions on CAT(0) cube complexes [GM] along with the malnormal combination theorem of Hsu and Wise [HW15b]. The need for the theory of train tracks (of free groups or free product automorphisms, see [BH92, FM15]) is limited to absolute train tracks for the fully irreducible case; it is encapsulated in the relative cubulation of free-product-by-cyclic groups [DM22].

For a hyperbolic group G possibly with torsion, if there exists a hyperbolic extension $G \rtimes \mathbb{Z}$, then G is virtually torsion-free (and residually finite) by Proposition 5.2. In particular, $G \rtimes \mathbb{Z}$ is virtually cubulable hyperbolic, and hence cubulable [Wis21, Lem. 7.14]. Agol's theorem [Ago13], therefore, implies that hyperbolic hyperbolic-by-cyclic groups are virtually (compact) special, and thus $\mathbb{Z}$-linear and their quasiconvex subgroups are separable [HW08].

We end this introduction with a question. Proposition 5.2 states that a hyperbolic group is virtually a free product of free and surface groups whenever it admits a hyperbolic automorphism. However, the converse is false as can be seen from a hyperbolic triangle group or the free product of two finite groups — these have finite outer automorphism groups.

Question. *Is there an algebraic characterisation of hyperbolic groups that admit hyperbolic automorphisms?*

Note that Pettet characterised virtually free groups with finite outer automorphism groups [Pet97].

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2 Free factor systems

A *free decomposition* of a group G is an isomorphism $G \cong A_1 * \cdots * A_k * F_r$, where $k \geqslant 0$, $r \geqslant 0$, each *free factor* A_i is not trivial, and F_r is free with rank r. We call $\mathcal{A} = (A_1, \ldots, A_k)$ a *free factor system* of G; it is *proper* unless $k = 1$ and $r = 0$. The integer $k + r$ is the *Kurosh co-rank* of the free factor system $\mathcal{A}$. A group is *freely indecomposable* if its free factor systems have Kurosh co-rank 1.

Assume G is finitely generated for the rest of this section. A *Grushko decomposition* of G is a free decomposition whose free factor system $\mathcal{A}$ has maximal Kurosh co-rank and free factors A_i are not $\mathbb{Z}$; in that case, we call $\mathcal{A}$ the *Grushko free factor system* and its Kurosh co-rank is the *Kurosh–Grushko rank* of G.

Recall the preorder of free factor systems of G: a free factor system $\mathcal{B} = (B_1, \ldots, B_q)$ is lower than $\mathcal{A}$ if each B_j is conjugate in G to a subgroup of some A_i. In this case, the free decomposition corresponding to $\mathcal{A}$ can be refined to that corresponding to $\mathcal{B}$ (as seen by the actions of A_i on $T_{\mathcal{B}}$, a Serre tree whose nontrivial vertex stabilisers are exactly the conjugates of the free factors B_j $(1 \leqslant j \leqslant q)$), and the Kurosh co-rank of $\mathcal{B}$ is at least that of $\mathcal{A}$ (see [DL22, Lemma 1.1]); if it is equal, then $\mathcal{A}$ is also lower than $\mathcal{B}$.

Let $\mathcal{A}$ be a free factor system of G. A *proper* $(G, \mathcal{A})$*-free factor* is a nontrivial point stabiliser of a nontrivial action of G on a tree, for which edge stabilizers are trivial, and in which the components of $\mathcal{A}$ are elliptic. In other words, it is a component of a free factor system that is higher than $\mathcal{A}$ in the preorder.

A minimal free factor system in this preorder is a Grushko free factor system; it is unique up to the preorder's equivalence relation. So any automorphism preserves the Grushko free factor system $\mathcal{A} = (A_1, \ldots, A_k)$, i.e.

it sends each A_i to a conjugate of some A_j. We say that a free factor system is *weakly invariant* with respect to $\phi \in \mathrm{Aut}(G)$ if some (positive) power of ϕ preserves it.

Lemma 2.1. *Suppose G is a finitely generated group. If $\mathcal{B} = (B_1, \ldots, B_\ell)$ is a proper free factor system, then each B_i has Kurosh–Grushko rank strictly lower than the Kurosh–Grushko rank of G.*

If G has Kurosh–Grushko rank $\geqslant 2$, then any automorphism $\phi\colon G \to G$ has a maximal weakly invariant proper free factor system.

Proof. Since $\mathcal{B}$ is proper, $G \cong B_i * H$ for some nontrivial group H. By uniqueness of the Grushko decomposition, the Kurosh–Grushko rank of G is the sum of those of B_i and H.

For the second assertion, as the Kurosh–Grushko rank is at least 2, the Grushko free factor system is proper and weakly invariant (with respect to ϕ). Restricting to weakly invariant proper free factor systems, any one with the lowest Kurosh co-rank is maximal in the preorder. □

3 Ingredients

Let G be a torsion-free group and $\psi\colon G \to G$ an automorphism. Suppose $\mathcal{B} = (B_1, \ldots, B_\ell)$ is a maximal weakly invariant proper free factor system (with respect to ψ) with Kurosh co-rank $\geqslant 3$; ψ preserves $\mathcal{B}$; and any ψ-periodic conjugacy class of nontrivial elements in G intersects some *peripheral* free factor B_i. These assumptions are equivalent to: $\mathcal{B}$ is a proper free factor system with Kurosh co-rank $\geqslant 3$; and $\psi \in \mathrm{Aut}(G, \mathcal{B})$ is both *relatively fully irreducible* and *relatively atoroidal* (relative to $\mathcal{B}$). For each B_i, let $k_i \geqslant 1$ be the smallest integer such that $\psi^{k_i}(B_i) = B_i^{g_i}$ for some $g_i \in G$. The *peripheral suspension* $B_i \rtimes \mathbb{Z}$ is the suspension of B_i by $\mathrm{ad}_{g_i} \circ \psi^{k_i}|_{B_i}$.

The first two named authors recently gave a *relative cubulation* of the mapping torus of a relatively fully irreducible relatively atoroidal automorphism. Their proof is adapted from Hagen and Wise's cubulation of hyperbolic irreducible free-by-cyclic groups [HW16].

Theorem 3.1 (cf. [DM22, Thm. 1.1])**.** *Under this section's assumptions, the mapping torus $G \rtimes_\psi \mathbb{Z}$ acts cocompactly on a CAT(0) cube complex, where each cell stabiliser is either trivial or conjugate to a finite index subgroup of some peripheral suspension $B_i \rtimes \mathbb{Z}$.* □

The cited theorem has an additional assumption (absence of twinned subgroups), but Guirardel remarked that this hypothesis is redundant. Two subgroups $H_1 \neq H_2$ of G are *twinned* in $\mathcal{B}$ if they are conjugates of some B_j, B_k and $\mathrm{ad}_g \circ \psi^n(H_i) = H_i$ $(i = 1, 2)$ for some $n \geqslant 1$ and $g \in G$.

Claim (Guirardel)**.** *As $\mathcal{B}$ has Kurosh co-rank $\geqslant 3$ and $\psi \in \mathrm{Aut}(G, \mathcal{B})$ is relatively fully irreducible, there are no twinned subgroups in $\mathcal{B}$.*

Our proof of the claim uses objects (expanding train tracks, limit trees, geometric trees of surface type) that we do not define here for the sake of brevity; we refer the reader to the relevant literature for each.

Proof. The automorphism ψ is represented by an expanding irreducible train track (see [DL22, Sec. 1.3]). Projectively iterating the train track produces the limit $(G, \mathcal{B})$-tree T and a ψ-equivariant expanding homothety $h \colon T \to T$ (see [BFH97, p. 232]). Note that nontrivial point stabilisers of T are ψ-periodic (up to conjugacy) by the finiteness of G-orbits of branch points in T [Hor17, Cor. 5.5] and the ψ-equivariance of h.

Suppose $H \leqslant G$ is a nontrivial nonperipheral point stabiliser of T. Then no proper $(G, \mathcal{B})$-free factor contains H — otherwise, the smallest such factor would be ψ-periodic, yet ψ is relatively fully irreducible. Thus T is geometric of surface type [Hor17, Sec. 6.2, Lem. 6.8] and the point stabiliser H is cyclic [Hor17, Prop. 6.10]. As H was arbitrary, all nonperipheral point stabilisers of T are cyclic; therefore, there are no twinned subgroups in $\mathcal{B}$ because they would generate a noncyclic nonperipheral T-elliptic subgroup by the ψ-equivariance of h. □

We will use the following theorem of Groves and Manning to upgrade relative cubulations in the next section.

Theorem 3.2 (cf. [GM, Thm. D])**.** *If a hyperbolic group Γ acts cocompactly on a CAT(0) cube complex so that cell stabilisers are quasiconvex and cubulable, then Γ is cubulable.* □

The cited theorem has "virtually special" in place of "cubulable". Since virtually cubulable hyperbolic groups are cubulable [Wis21, Lem. 7.14], the properties "virtually special" and "cubulable" are equivalent for hyperbolic groups by Agol's theorem [Ago13]. In particular, for hyperbolic groups, being cubulable is a commensurability invariant.

Finally, for sporadic cases when the Kurosh co-rank is 2, we will need a specialisation of Hsu and Wise's malnormal combination theorem:

Theorem 3.3 (cf. [HW15b, Cor. C])**.** *Suppose $\Gamma = \Gamma_1 *_{\langle c \rangle} \Gamma_2$ or $\Gamma_1 *_{\langle d \rangle^s = \langle c \rangle}$ is hyperbolic and $\langle c \rangle$ is an infinite cyclic malnormal subgroup of Γ. If each Γ_i is cubulable, then Γ is cubulable.* □

The two decompositions can be stated together as: "Γ splits over $\langle c \rangle$."

4 The bootstrap

The following proposition is due to Sela (see Proposition 5.1 for a proof).

Proposition 4.1 (cf. [Sel97, Cor. 1.10])**.** *Assume G is a torsion-free hyperbolic group and some extension $G \rtimes_\phi \mathbb{Z}$ does not contain a copy of $\mathbb{Z}^2$. If G is freely indecomposable, then it is the fundamental group of a closed surface.* □

We may now prove the central result of this note:

Theorem 4.2. *Let G be a torsion-free hyperbolic group. If $G \rtimes_\phi \mathbb{Z}$ is hyperbolic, then it is cubulable.*

Proof. We proceed by induction on the Kurosh–Grushko rank.

If the Kurosh–Grushko rank of G is 1, then G is freely indecomposable. By Proposition 4.1, G is a closed surface group and, by the classification of its automorphisms, ϕ is pseudo-Anosov. Then $G \rtimes_\phi \mathbb{Z}$ is famously the fundamental group of a closed hyperbolic 3-manifold and cubulable, as already mentioned in Section 1. Assume $n \geqslant 2$ and the theorem holds for torsion-free hyperbolic groups of Kurosh–Grushko rank $< n$.

Let the Kurosh–Grushko rank of G be n. Lemma 2.1 provides a maximal weakly invariant proper free factor system $\mathcal{B} = (B_1, \ldots, B_\ell)$, and each B_i has Kurosh–Grushko rank $< n$. Some power ψ of ϕ preserves $\mathcal{B}$, i.e. $\psi \in \mathrm{Aut}\,(G, \mathcal{B})$ is relatively fully irreducible. As the peripheral free factors B_i are quasiconvex in G, one can show that $G \rtimes_\psi \mathbb{Z}$ admits Lipschitz retractions to the peripheral suspensions $B_i \rtimes \mathbb{Z}$; hence the latter are quasiconvex and hyperbolic. By the induction hypothesis, each $B_i \rtimes \mathbb{Z}$ is cubulable.

We distinguish two cases. The first case is when the Kurosh co-rank of $\mathcal{B}$ is at least 3. Since ψ is hyperbolic, it is also relatively atoroidal. By Theorem 3.1, $G \rtimes_\psi \mathbb{Z}$ acts cocompactly on a CAT(0) cube complex, where each cell stabiliser is either trivial or conjugate to a finite index subgroup of some quasiconvex cubulable $B_i \rtimes \mathbb{Z}$. Groves and Manning's Theorem 3.2 thus implies that $G \rtimes_\psi \mathbb{Z}$ is cubulable. It has finite index in $G \rtimes_\phi \mathbb{Z}$, so the latter is also cubulable.

The last case is when the Kurosh co-rank of $\mathcal{B}$ is 2. There are three possibilities: G is $B_1 * B_2$, $B_1 * F_1$, or F_2. We rule out the third possibility as $F_2 \rtimes \mathbb{Z}$ is never hyperbolic. To conclude the proof, we will prove that $\Gamma = G \rtimes_\psi \langle c \rangle$ (virtually) satisfies the hypotheses of Hsu and Wise's Theorem 3.3, and hence is cubulable. Note that $\langle c \rangle$ is a maximal cyclic subgroup of Γ, and hence malnormal. It remains to show that Γ splits over $\langle c \rangle$.

In the first possibility, up to taking the square of ψ, we may assume that ψ preserves the conjugacy class of both B_1 and B_2. After conjugation (which does not change the mapping torus), we may assume it fixes B_1 (setwise) and, being an automorphism, it sends B_2 to a conjugate by an element of B_1. After further conjugation, it fixes both B_1 and B_2. Then the mapping torus $\Gamma = (B_1 * B_2) \rtimes_\psi \langle c \rangle \cong (B_1 \rtimes \langle c \rangle) *_{\langle c \rangle} (B_2 \rtimes \langle c \rangle)$.

In the second possibility, we write $G = B_1 * \langle s \rangle$. Up to taking the square of ψ and composing with a conjugation, we may assume that $\psi(B_1) = B_1$ and $\psi(s) = bs$ for some $b \in B_1$. Consider $G \rtimes_\psi \langle c \rangle$, where one has the relation $c^{-1}sc = bs$, or written differently $s^{-1}cbs = c$. Then, rewriting the presentation, one has that

$$\Gamma = (B_1 * \langle s \rangle) \rtimes_\psi \langle c \rangle \cong (B_1 \rtimes \langle c \rangle)*_{\langle cb \rangle^s = \langle c \rangle}$$

where the last operation is an HNN extension with the stable letter s that conjugates $\langle cb \rangle$ to $\langle c \rangle$ (and actually cb to c). $\square$

5 Once more, with torsion

Now G is a finitely presented group (possibly with torsion). It has a maximal decomposition as the fundamental group of a finite graph of groups with finite edge groups [Dun85]. The infinite vertex groups are thus one-ended [Sta71]. We call this a *Dunwoody–Stallings decomposition*. It is not unique, but the conjugacy classes of infinite vertex groups are uniquely defined: they are conjugacy classes of the maximal one-ended subgroups of G. The following is a generalisation of Proposition 4.1.

Proposition 5.1. *Assume G is a hyperbolic group (possibly with torsion) and some extension $G \rtimes_\phi \mathbb{Z}$ does not contain a copy of $\mathbb{Z}^2$. Then every maximal one-ended subgroup of G is virtually a closed surface group.*

Proof. Let H be a maximal one-ended subgroup of G. Since there are only finitely many conjugacy classes of such subgroups, there exists an integer $k \geqslant 1$ and an element $g \in G$ such that $\psi = (\mathrm{ad}_g \circ \phi^k)|_H$ fixes H setwise.

The group $H \rtimes_{\psi} \mathbb{Z}$ embeds in $G \rtimes_{\phi} \mathbb{Z}$. Since H is one-ended, its JSJ decomposition is preserved by ψ [Bow98]. The lack of $\mathbb{Z}^2$ in $G \rtimes_{\phi} \mathbb{Z}$ imposes that the JSJ is trivial but not a rigid vertex [Pau91]. It is therefore a vertex of surface type. In particular, H is virtually a closed surface group (see, for instance, [Mar07, Sec. 4]). □

We are now ready to state the main observation of this section.

Proposition 5.2. *If G is a hyperbolic group (possibly with torsion) and some extension $G \rtimes_{\phi} \mathbb{Z}$ does not contain a copy of $\mathbb{Z}^2$, then G has a characteristic finite index subgroup that is a free product of closed surface groups and free groups. In particular, it is residually finite.*

Proof. Let $\mathbb{X}$ be a Dunwoody–Stallings decomposition of G. Recall that $\mathbb{X}$ has underlying finite graph X with vertex groups $\mathbb{X}_v$ ($v \in X^{(0)}$) and finite edge groups $\mathbb{X}_e$ ($e \in X^{(1)}$). For each vertex v, denote by H_v a normal finite index subgroup of $\mathbb{X}_v$ that is either trivial or a closed surface group, as guaranteed by Proposition 5.1.

As the subgroups H_v are torsion-free, the surjections $q_v \colon \mathbb{X}_v \to \mathbb{X}_v / H_v$ are injective on finite subgroups. Thus we define a graph of finite groups $\mathbb{Y}$ with underlying graph X, vertex groups $\mathbb{X}_v / H_v$, and edge groups $\mathbb{X}_e$; the surjections q_v induce a surjection $q \colon G \to \pi_1(\mathbb{Y})$ with a torsion-free kernel. The quotient $\pi_1(\mathbb{Y})$ is virtually free by Karrass, Pietrowski, and Solitar's characterisation [KPS73, Thm. 1].

Let $J \leqslant \pi_1(\mathbb{Y})$ be a free finite index subgroup. Since J and the kernel of q are torsion-free, the preimage $q^{-1}(J) \leqslant G$ is a torsion-free finite index subgroup. The intersection H of subgroups of G with index $[G : q^{-1}(J)]$ is a characteristic torsion-free finite index subgroup. The decomposition $\mathbb{X}$ of G induces a Grushko decomposition of H whose freely indecomposable free factors are closed surface groups. □

We may extend Brinkmann's thesis [Bri00] to the case with torsion.

Corollary 5.3. *Suppose G is a hyperbolic group. Then $G \rtimes_{\phi} \mathbb{Z}$ is hyperbolic if and only if it does not contain a copy of $\mathbb{Z}^2$.*

The forward implication is standard. Conversely, if $G \rtimes_{\phi} \mathbb{Z}$ does not contain a copy of $\mathbb{Z}^2$, then the same holds for the finite index subgroup $G_0 \rtimes_{\phi|_{G_0}} \mathbb{Z}$, where G_0 is the torsion-free subgroup given by Proposition 5.2. As $G_0 \rtimes \mathbb{Z}$ is hyperbolic [Bri00], so is $G \rtimes \mathbb{Z}$.

Corollary 5.4. *If G and $G \rtimes_\phi \mathbb{Z}$ are hyperbolic groups, then $G \rtimes_\phi \mathbb{Z}$ is cubulable.*

Again, we take the torsion-free subgroup G_0 given by Proposition 5.2 and obtain the finite index subgroup $G_0 \rtimes_{\phi|_{G_0}} \mathbb{Z}$ of $G \rtimes_\phi \mathbb{Z}$. This subgroup is cubulable by Theorem 4.2, and hence the latter is too.

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