HYPERBOLIC IMMERSIONS OF FREE GROUPS

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ABSTRACT. We prove that the mapping torus of a graph immersion has a word-hyperbolic fundamental group if and only if the corresponding endomorphism does not produce Baumslag-Solitar subgroups. Due to a result by Reynolds, this theorem applies to all injective endomorphisms of F_2 and nonsurjective fully irreducible endomorphisms of F_n. We also give a framework for extending the theorem to all injective endomorphisms of F_n.

1. INTRODUCTION

Thurston [15] proved that the interior of a *mapping torus* M_f of a hyperbolic surface homeomorphism $f : S \to S$ has a finite volume hyperbolic structure if and only if the homeomorphism is isotopic to a *pseudo-Anosov* homeomorphism, which by Nielsen-Thurston Classification is equivalent to saying f has no periodic homotopy classes of essential simple closed curves. Assuming S is closed, Thurston's result states, in particular, that f is isotopic to a pseudo-Anosov homeomorphism if and only if $\pi_1(M_f)$ is *word-hyperbolic*, i.e., its Cayley complex satisfies a linear isoperimetric inequality [1, 9].

In the spirit of Thurston's result, Bestvina-Feighn proved that when G is word-hyperbolic and $\phi : G \to G$ is a *hyperbolic* automorphism, then $G \rtimes_\phi \mathbb{Z}$ is word-hyperbolic. Hyperbolic automorphisms are defined in Section 6. Peter Brinkmann later proved that *atoroidal* automorphisms of free groups, i.e., automorphisms with no nontrivial periodic conjugacy classes, are hyperbolic. Together, these theorems give:

Theorem (Bestvina-Feighn [2], Brinkmann [4])**.** *Let $\phi : F \to F$ be an automorphism of a free group of finite rank. The following are equivalent:*

(1) *ϕ is hyperbolic.*
(2) *$F \rtimes_\phi \mathbb{Z}$ is word-hyperbolic.*
(3) *$F \rtimes_\phi \mathbb{Z}$ has no $\mathbb{Z}^2$ subgroup.*
(4) *ϕ is atoroidal.*

One might now ask if a similar statement is true when ϕ is injective but not surjective. In this case, we can no longer form a semi-direct product $F \rtimes \mathbb{Z}$ but the same presentation defines an *ascending HNN-extension* denoted by $F*_\phi$ (Section 2). Unlike automorphisms, an iterate of ϕ may now map a nontrivial element to a conjugate of some proper power.

Example 1.1. Let $F_1 \cong \mathbb{Z}$ be generated by c and let d be any positive integer, then we can define an endomorphism $\psi : F_1 \to F_1$ by $\psi(c) = c^d$. We denote $\mathbb{Z}*_\psi$ by $BS(1, d)$ and these are the **metabelian Baumslag-Solitar (BS)** groups. For the rest of this paper, we will explore $BS(1, 2)$.

Baumslag-Solitar groups are obstructions to word-hyperbolicity and our main theorem states that, under certain hypothesis, these are the only obstructions.

Theorem 6.7. *Suppose $f : \Gamma \to \Gamma$ is an immersion of a finite graph. The following are equivalent:*

(1) *$\pi_1(M_f)$ is word-hyperbolic.*
(2) *$\pi_1(M_f)$ contains no $BS(1, d)$ subgroups for $d \geq 1$.*
(3) *There are no $k, d \geq 1$ and nontrivial loop σ in Γ such that $f^k(\sigma)$ is freely homotopic to σ^d.*

While proving this theorem, we will derive one more equivalent condition that we currently omit until relevant definitions are given in Section 3. Our theorem generalizes theorems by Ilya Kapovich [10, Theorem 5.5] and François Gautero [8, Theorem 13.2]: Kapovich assumed $\phi(F)$ was an *immersed subgroup* (equivalently, f is an immersion on the rose) and the proof was algebraic; Gautero assumed ϕ was *hyperbolic* with malnormal image and their argument was topological. We give a topological proof along the lines of Kapovich which allowed us to generalize both results. The extra condition mentioned before also gives us an algorithm that determines whether Condition (3) holds for a given immersion (Remark 3.12, Corollary 6.8). Even though the hypothesis of Theorem 6.7 is restrictive, using a theorem of Patrick Reynolds [14], we show that it applies to all *fully irreducible* endomorphisms of F_n and all injective endomorphisms of F_2:

Corollary 7.3. *Let $\phi : F_n \to F_n$ be a fully irreducible endomorphism. The following are equivalent:*

(1) *$F_n *_\phi$ is word-hyperbolic.*
(2) *$F_n *_\phi$ has no $BS(1, d)$ subgroups for $d \geq 1$.*
(3) *There are no $k, d \geq 1$ and $1 \neq g \in F_n$ such that $\phi^k(g)$ is conjugate to g^d in F_n.*

Corollary 7.4. *Let $\phi : F_2 \to F_2$ be an injective endomorphism. The following are equivalent:*

(1) *$F_2 *_\phi$ is word-hyperbolic.*
(2) *$F_2 *_\phi$ has no $BS(1, d)$ subgroups for $d \geq 1$.*
(3) *There are no $k, d \geq 1$ and $1 \neq g \in F_2$ such that $\phi^k(g)$ is conjugate to g^d in F_n.*

Example 1.2 (Sapir Group)**.** Let F_2 be a free group generated by a, b and $\varphi : F_2 \to F_2$ be given by $\varphi(a) = ab$ and $\varphi(b) = ba$. The **Sapir Group** is the asc. HNN-ext. $F_2 *_\varphi$. Since $\varphi(F_2)$ is an immersed subgroup, one can use Kapovich's result to prove the group is word-hyperbolic. The only proof we have found is due to J. O. Button [6, Theorem 4.1] who used the action on cyclically reduced words to directly show that no iterate of φ maps a nontrivial element to a conjugate of its power. We will provide a topological proof of this fact. (Example 6.6)

The two examples given in this section, ψ and φ, will be used throughout the paper to illustrate the various ideas involved and three new examples will be given in Section 7. The reader is encouraged to choose a random nonsurjective injective endomorphism of

F_3 to work with as another example. If the endomorphism is not induced by any graph immersion, this would be a useful example to have when generalizing Corollary 7.4 to F_n.

We now briefly sketch the proof of Theorem 6.7. The first implication, $(1) \implies (2)$, is the fact that BS subgroups are obstructions to word-hyperbolicity: word-hyperbolic groups have virtually cyclic centralizers and their cyclic subgroups are quasi-isometrically embedded [1]. Kapovich proved $(2) \implies (3)$ [10, Lemma 2.3]. We prove $(3) \implies (1)$ using the Bestvina-Feighn combination theorem. Briefly, the combination theorem states that if all annuli $\alpha : S^1 \times I \to M_f$ (with some technical conditions) have *uniform exponential growth*, then $\pi_1(M_f)$ is word-hyperbolic. It remains to show that all annuli do have uniform exponential growth.

Informally, the growth of an annulus $g(\alpha)$ is the ratio of the larger end of the annulus to the center of the annulus and the length $l(\alpha)$ is the distance between the ends. The annuli have uniform exponential growth if there is $\lambda > 1$ such that $\lambda^{l(\alpha)} \leq g(\alpha)$ for long enough annuli.

Lift α to the natural infinite cyclic cover of M_f to get $\tilde{\alpha} : S^1 \times I \to \tilde{M}_{\mathbb{Z}}$. This cover has a preferred direction in which things grow exponentially due to the atoroidal and immersion assumption on f. In particular, if $\tilde{\alpha}$ is monotone with respect to this direction, then it will have uniform exponential growth along this direction. So we need to show that those that fail to be monotone have uniform exponential growth as well. The definition of annuli forces such $\tilde{\alpha}$ to consist of two segments: one increasing, one decreasing.

Proposition 5.3 uses the lack of f-invariant loops and immersion assumption on f to show that, for all non-monotone $\tilde{\alpha}$, we can assume that the decreasing segments have uniformly bounded lengths. So long enough annuli will have negligible decreasing segments and behave like monotone annuli, i.e., they have uniform exponential growth. This concludes the sketch proof. Note that when ϕ is an automorphism, all annuli will be monotone but the uniform exponential growth is difficult to establish since f is not an immersion. The second case where an annulus fails to be monotone is a new phenomenon unique to nonsurjective endomorphisms.

To generalize Theorem 6.7 to all π_1-injective graph maps, it would be helpful to have an analogue for hyperbolic automorphisms. In Section 6, we define *hyperbolic* graph maps and give a sufficient condition for when the fundamental groups of their mapping tori are word-hyperbolic. This reduces the general problem to showing an analogue of Brinkmann's theorem and generalizing Proposition 3.11.

Overview of the paper: For the most part, we follow the structure of Kapovich's paper [10]. In Section 2, we set the assumptions, definitions, and notations that will be used throughout the paper. In Section 3, we introduce the pullback of a graph immersion and we prove a pullback stabilizing proposition (Proposition 3.11). In Section 4, annuli and other relevant definitions are given and the combination theorem is stated. Section 5 is the crucial bridge between the previous two sections as we interpret Proposition 3.11 in

terms of annuli (Proposition 5.3). In Section 6, we use this annuli interpretation and the combination theorem to prove word-hyperbolicity. In the process, we define hyperbolic graph maps and mention how this may help extend the main theorem from immersions to all π_1-injective graph maps. Section 7 contains the application of the main theorem to fully irreducible endomorphisms of F_n and all injective endomorphisms of F_2.

2. Definitions and Notations

We define finite graphs to be finite 1-dimensional CW-complexes. The 0-cells are called the vertices and the 1-cells are edges. A **core graph** is a finite graph with no valence-1 vertices. In this section and the remainder of the paper, Γ is a connected core graph whose vertices have valency ≥ 3.

A (topological) map $f : \Gamma \to \Gamma$ is an **immersion** if it is a locally injective map. The valency restriction on Γ is only included as it implies that immersions map vertices to vertices. The restriction can be removed if, additionally, immersions are assumed to map vertices to vertices. The **mapping torus of** f, M_f, is defined to be

$$\left(\Gamma \times [-1/2, 1/2]\right)/\sim \text{ with the relation } (x, 1/2) \sim (f(x), -1/2) \; \forall x \in \Gamma$$

We also set, for this section and the remainder of the paper, $S^1 = \mathbb{R}/\mathbb{Z}$ and, for any interval $I \subset \mathbb{R}$, $I_{\mathbb{Z}} = I \cap \mathbb{Z}$. The **edge space** is the integer cross-section $\Gamma \times \{0\}$ while the **vertex space** is the complement in M_f of the edge space. The interval used in our definition for M_f is not standard but it has been chosen so that the edge space, which is the space we will be most interested in, lies in the integer cross-section of M_f. This was a purely aesthetic choice.

An immersion $f : \Gamma \to \Gamma$ induces an injective (outer) endomorphism $\phi = f_*$ of $F = \pi_1(\Gamma)$, well-defined up to post-composition with an inner automorphism. By Van Kampen's theorem:

$$\pi_1(M_f) \cong \langle\, F, t \mid t^{-1}xt = \phi(x), \forall x \in F \,\rangle$$

When ϕ is an automorphism, the latter is the presentation of the semi-direct product $F \rtimes_\phi \mathbb{Z}$. For an injective (not necessarily surjective) endomorphism ϕ, it is the presentation of its **ascending HNN-extension** and denoted by $F*_\phi$.

A map $f : \Gamma \to \Gamma$ has **an invariant loop** if there exists positive integers k, d, and a nontrivial loop σ in Γ such that $f^k(\sigma) \simeq \sigma^d$, i.e., $f^k(\sigma)$ is freely homotopic to σ^d. Equivalently, for the induced endomorphism $\phi = f_*$, there are $k, d \geq 1$ and a nontrivial $g \in F$ such that $[\phi^k(g)] = [g^d]$, i.e., $\phi^k(g) = xg^dx^{-1}$ for some $x \in F$. We shall refer to d as the **degree of the invariant loop**.

Example 2.1 (Continuing Example 1.1)**.** Let $G = S^1$. We can induce ψ with the map $g : G \to G$ given by $x \mapsto 2x$. This is an immersion on the circle. As mentioned earlier, $\pi_1(M_g) \cong BS(1, 2)$ is not word-hyperbolic. Clearly, the graph G is a g-invariant loop with degree $d = 2$.

Example 2.2 (Continuing Example 1.2)**.** Let H be two copies of S^1 with their basepoints $0 \in S^1$ identified and label the copies a and b respectively. Then we can induce φ with an immersion $h : H \to H$ that maps a onto the path ab and b onto ba. We shall eventually show that $\pi_1(M_h)$ is word-hyperbolic and h has no invariant loop.

3. Pullbacks

Definition 3.1. The **pullback** or **fibered product** of graph immersions $g : A \to G$ and $h : B \to G$ is the topological space:
$$A \times_G B = \{ (x, y) \in A \times B \; : \; g(x) = h(y) \}$$
For $i \geq 1$, let Γ_i be the pullback of f^i and f^i. Set $\Gamma_0 = \{(x, y) \in \Gamma \times \Gamma \; : \; x = y\}$.

It follows from the local injectivity of an immersion $f : \Gamma \to \Gamma$ that each Γ_i is a finite graph and $\Gamma \cong \Gamma_0 \subset \Gamma_1 \subset \Gamma_2 \subset \cdots \subset \Gamma \times \Gamma$.

Definition 3.2. For any graph X, a **direction at a point** $x \in X$ is an end of $X - x$. We abuse notation and denote the set of directions at x by $T_x X$. Given a topological map $q : X \to X$ that preserves vertices and is locally injective on interior of edges, the **derivative map** of q at x is the induced map $dq_x : T_x X \to T_{q(x)} X$.

Note that the valency at x is the number of directions at x and all derivative maps are injective if and only if the graph map is an immersion. The definition is given here rather than the previous section since it will only be used in the following lemma and Lemma 6.3.

Lemma 3.3. *Let $f : \Gamma \to \Gamma$ be an immersion. Then for all $i \geq 0$, Γ_i is a union of some components of Γ_{i+1}.*

Proof. Fix some $i \geq 0$ and let C and C' be components of Γ_i and Γ_{i+1} resp. such that $C \subset C'$. We want to show that $C = C'$.

For any $p \in C$, let $v_p = C$-valency of p, i.e., the number of ends of $C - p$, and similarly define $v'_p = C'$-valency of p. Recall p corresponds to two points $p_1, p_2 \in \Gamma$ such that $f^i(p_1) = f^i(p_2)$ and let $q = f^i(p_1)$. The C-valency of p corresponds to a maximal list of pairs of directions $(a_1, \alpha_1), \ldots, (a_{v_p}, \alpha_{v_p})$ based at p_1 and p_2 resp. such that $df^i_{p_1}(a_j) = df^i_{p_2}(\alpha_j)$. Suppose $v_p < v'_p$, then that means there is at least one extra pair of directions (b, β) such that $df^i_{p_1}(b) \neq df^i_{p_2}(\beta)$ but $df^{i+1}_{p_1}(b) = df^{i+1}_{p_2}(\beta)$. But this means that $q = f^i(p_1) = f^i(p_2)$ has two distinct directions $df^i_{p_1}(b), df^i_{p_2}(\beta)$ that map to the same direction under the derivative map df_q. This contradicts the fact f is an immersion. Therefore, $v_p = v'_p$ and thus $C = C'$ as all points of C have the same valency in both C and C'. $\square$

For this section and the remaining sections, set $\hat{\Gamma}_i$ to be the maximal core subgraph of $\Gamma_i - \Gamma_{i-1}$. There is a natural immersion $\hat{f} : \Gamma_i \to \Gamma_{i-1}$ given by $\hat{f}(x, y) = (f(x), f(y))$ which restricts to an immersion $\hat{\Gamma}_i \to \hat{\Gamma}_{i-1}$.

Lemma 3.4. *Let $f : \Gamma \to \Gamma$ be an immersion. If $\hat{\Gamma}_i$ is empty, then so are all $\hat{\Gamma}_j$ for $j > i$. If $\hat{\Gamma}_i$ consists of loops and $\hat{\Gamma}_{i+1}$ is nonempty, then $\hat{\Gamma}_{i+1}$ consists of loops too.*

Proof. Since we have the immersion $\hat{f} : \hat{\Gamma}_i \to \hat{\Gamma}_{i-1}$ for all $i \geq 1$, if $\hat{\Gamma}_i$ is empty, then so are all $\hat{\Gamma}_j$ for $j > i$. Furthermore, the only core graph that immerses into a loop is a loop itself. So if $\hat{\Gamma}_i$ consists of disjoint loops and $\hat{\Gamma}_{i+1}$ is nonempty, then $\hat{\Gamma}_{i+1}$ consists of disjoint loops. $\square$

We say that the **pullbacks stabilize** if $\hat{\Gamma}_i = \emptyset$ for some i.

Example 3.5 (Continuing Example 2.1). For $g : G \to G$, the pullback G_1 is two disjoint loops: one is the diagonal $G_0 \cong G$ and the other is $\hat{G}_1 = \left\{ (x, y) \in S^1 \times S^1 : y - x = 1/2 \right\}$. More generally,

$$\hat{G}_i = \left\{ (x, y) \in S^1 \times S^1 : y - x = \frac{(2j - 1)}{2^i}, \ j \in [1, 2^{i-1}]_{\mathbb{Z}} \right\}$$

Topologically, the components for all G_i are loops and the number of components in $\hat{G}_i$ doubles with each iteration. The picture on the left in Figure 1 shows the first two pullbacks.

Example 3.6 (Continuing Example 2.2). For $h : H \to H$, the pullback H_1 consists of a copy of H and two extra loops. The second pullback H_2 consists of H_1 and contractible components. Therefore, the core subgraph $\hat{H}_2$ and all subsequent $\hat{H}_i$ are empty and so the pullbacks stabilize. Contrast this behavior with that of g in previous example. Proposition 3.11 below states that pullbacks stabilize if the immersion has no invariant loops. The picture on the right in Figure 1 illustrates H_1.
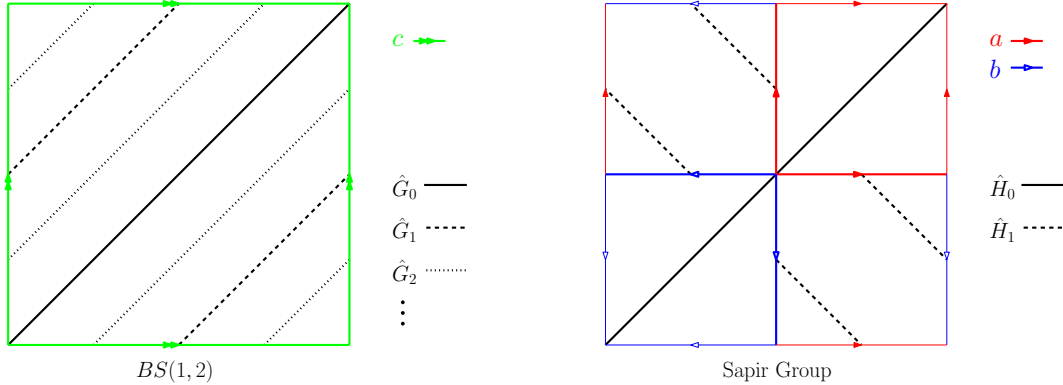

FIGURE 1. Pullbacks for our main examples.

Walter Neumann used pullbacks to generalize a result by Hanna Neumann [12]:

Theorem 3.7 ([13, Proposition 2.1]). *Let $M, N \leq F$ be nontrivial finitely generated subgroups. Then*

$$\sum_{[[g]] \in M \backslash F / N} \max(0, \operatorname{rank}(M \cap gNg^{-1}) - 1) \leq 2(\operatorname{rank}(M) - 1)(\operatorname{rank}(N) - 1)$$

where the sum ranges over all (M, N)-double cosets $[[g]] = MgN$.

Remark 3.8. The *strengthened Hanna Neumann conjecture* is that the 2 can be dropped and it has been proven (Friedman [7], Mineyev [11]) but, for our purporses, Walter Neumann's bound will suffice.

Lemma 3.9. *Let $f : \Gamma \to \Gamma$ be an immersion and $M = 2(\operatorname{rank}(\Gamma) - 1)^2$. For $i \geq M$, the graphs $\hat{\Gamma}_i$ are either all disjoint loops or eventually empty.*

Proof. Let $\phi = f_* : F \to F$ be the induced endomorphisms. The components of Γ_i correspond to the intersections $\phi^i(F) \cap g\phi^i(F)g^{-1}$ as $[[g]]$ ranges over $\phi^i(F)$-double cosets. Hence, we have the following inequality for all i:

$$\chi^-(\Gamma_i) = \sum_{[[g]]} \max(0, \operatorname{rank}(\phi^i(F) \cap g\phi^i(F)g^{-1}) - 1) \leq 2(\operatorname{rank}(\phi^i(F)) - 1)^2 = M$$

where χ^- is the negative Euler characteristic of the maximal core subgraph and the sum ranges over all $\phi^i(F)$-double cosets $[[g]]$. The chain $\Gamma_0 \subset \Gamma_1 \subset \cdots$ and Lemma 3.4 imply that the nondecreasing sequence $(\chi^-(\Gamma_i))_{i=0}^\infty$ becomes constant once two consecutive terms are equal. Since the sequence is bounded by M, the subsequence $(\chi^-(\Gamma_i))_{i=M}^\infty$ must be constant and, by Lemma 3.4 again, the graphs $\hat{\Gamma}_i$ are disjoint loops for all $i \geq M$ or are eventually empty. $\qquad\square$

Example 3.10. For the immersion g in Example 3.5, we get $M_1 = 2(1 - 1)^2 = 0$. So we conclude from the bound that the graphs $\hat{G}_i$ $(i \geq 0)$ are either all disjoint loops or eventually empty. As we verified earlier, it is the former case.

In Example 3.6, we have $M_2 = 2(2 - 1)^2 = 2$. So $\hat{H}_i$ $(i \geq 2)$ are either all disjoint loops or eventually empty. We verified earlier that it is the latter case.

Proposition 3.11. *Suppose $f : \Gamma \to \Gamma$ is an immersion and $\hat{\Gamma}_i$ is nonempty for all i. Then f has an invariant loop with degree $d \geq 2$.*

Proof. Suppose $\hat{\Gamma}_i$ is nonempty for all i. Let $(\gamma_k)_{k=0}^\infty$ and $(m_k)_{k=0}^\infty$ be sequences of immersed loops and strictly increasing positive integers such that $\gamma_k \subset \hat{\Gamma}_{m_k}$. Lemma 3.9 gives an explicit M such that, for all $m \geq M$, the components of $\hat{\Gamma}_m$ are loops. Passing to subsequence if necessary, assume that $m_k > M$ for all k and the sequence of loops $(\hat{f}^{m_k - M}(\gamma_k))$ in $\hat{\Gamma}_M$ all lie in the same component, C; this can be done as $\hat{\Gamma}_M$ is finite. Therefore, for all k, there is a nonzero integer n_k satisfying $\hat{f}^{m_k - M}(\gamma_k) \simeq C^{n_k}$. The component C is a pair of loops $(c_-, c_+) \subset \Gamma \times \Gamma$ and, similarly, γ_k is a pair of loops (σ_{-k}, σ_k) so that we have $f^{m_k - M}(\sigma_{\pm k}) \simeq c_\pm^{n_k}$. We are using $\pm$ signs to distinguish the left and right factors.

Let $Z \subset \Gamma$ denote the subgraph of non-expanding edges, i.e., edges whose images under f-iteration have uniformly bounded combinatorial lengths. Then f-orbits of edges of Z

are eventually periodic. In particular, for large enough k, if $x, y \in Z$ and $f^k(x) = f^k(y)$ then $f^{k-1}(x) = f^{k-1}(y)$. So for large enough k, $(x, y) \in \hat{\Gamma}_k$ implies at least one of x or y is not in Z, i.e., at least one of x or y is a point in an expanding edge. By passing to a subsequence if necessary, we can assume without loss of generality that all loops σ_k share an expanding edge $e \subset \Gamma$.

Set c to be the minimal root of c_+ and $N = N(c)$ to be the number of subpaths of c that are also loops (subloops of c).

Since f is an immersion, e is an expanding edge of σ_k for all k, and $f^{m_k-M}(\sigma_k) \simeq c_+^{n_k}$, we have $f^i(e)$ surjects onto c for large enough i. In fact, it will surject onto any power of c for large enough i. Fix a large k and a subsegment $s \subset e$ such that $f^{m_k-M}(s) = c^{N+1}$. Let $m = m_{k+1} - m_k$, then $f^{m_{k+1}-M}(s) = f^m(f^{m_k-M}(s)) = f^m(c^{N+1})$ is a subloop of $f^{m_{k+1}-M}(\sigma_{k+1}) \simeq c^{n_{k+1}}$. In fact, for all $1 \le j \le N+1$, $f^m(c^j)$ is a subloop of $c^{n_{k+1}}$. Thus, there is a subloop $\epsilon_j \subset c$ and a positive integer s_j such that

$$f^m(c^j)\epsilon_j \simeq c^{s_j}$$

By the choice of N, there is a pair $1 \le t < t' \le N+1$ such that $\epsilon_t = \epsilon_{t'} = \epsilon$. Therefore,

$$f^m(c^t)\epsilon \simeq c^{s_t}, \quad f^m(c^{t'})\epsilon \simeq c^{s_{t'}}$$

and

$$f^m(c)^{t'-t} = f^m(c^{t'-t}) \simeq c^{s_{t'}-s_t}$$

We also have the following inequalities: $m = m_{k+1} - m_k > 0$, $t' - t > 0$, and $s_{t'} - s_t > 0$. As c is a minimal root, the last equation implies $f^m(c) \simeq c^d$ for some $d \ge 1$; in fact, $d \ge 2$ since $e \subset c$ is expanding. $\qquad\square$

Remark 3.12. Using the pigeonhole principle, there is a recipe for a number $L(f)$ such that if the original sequence (m_k) had $L(f)$ entries instead of being infinite, then the necessary subsequences can still be taken producing a nonempty subsequence. In particular, $\hat{\Gamma}_{L(f)} \ne \emptyset$ implies f has invariant loop with degree $d \ge 2$. This recipe uses the following computable numbers: $M = 2(\mathrm{rank}(\Gamma)-1)^2$; the number of components of $\hat{\Gamma}_M$; the number of edges in Γ; the maximum $N(c_-)$ or $N(c_+)$ as $C = (c_-, c_+)$ ranges over all components of $\hat{\Gamma}_M$; and the maximum combinatorial length of c_- or c_+ as C ranges over all components of $\hat{\Gamma}_M$.

4. The Combination Theorem

The Bestvina-Feighn combination theorem gives us a sufficient condition for when mapping tori of free groups, and more generally graphs of δ-hyperbolic spaces, have word-hyperbolic fundamental groups. We need to define some terms before stating it.

Definition 4.1. Let m be a positive integer. An **annulus of length** $2m$ is a map $\alpha : S^1 \times [-m, m] \to M_f$ satisfying the following conditions:

(1) It is transverse to the edge space.
(2) The α-preimage of the edge space is $S^1 \times [-m, m]_{\mathbb{Z}}$.

(3) For $i \in [-m, m]_{\mathbb{Z}}$, $\alpha|_{S^1 \times \{i\}}$ is locally injective everywhere with exception possibly at the basepoint $0 \in S^1 \cong S^1 \times \{i\}$.

(4) For $i \in [-m, m-1]_{\mathbb{Z}}$, $\alpha|_{\{0\} \times [i,i+1]}$ is not homotopic rel endpoints into the edge space

For simplicity, set $\alpha_t = \alpha|_{S^1 \times \{t\}}:S^1 \to M_f$ and $\alpha^* = \alpha|_{\{0\} \times [-m,m]}:[-m, m] \to M_f$.

When t is an integer, we will refer to the based loops α_t as the **rings** of the annulus; the path α^* will be **trace** of the basepoint. Informally, the rings will be used to define the thickness of the annulus while the pieces of the trace in the vertex space will define the distance between consecutive rings of the annulus.

The **girth** of α is $l(\alpha_0)$ where l is the (combinatorial) length measured after tightening rel. basepoint in the edge space. For $\lambda > 1$, we say that α is λ-**hyperbolic** if

$$\lambda l(\alpha_0) \leq \max\{l(\alpha_{-m}), l(\alpha_m)\}$$

For every $i \in [-m, m-1]_{\mathbb{Z}}$, define $\tau_i : (i, i+1) \to \Gamma$ by

$$\tau_i(t) = \begin{cases} x & \text{if } \alpha^*(t) = (x, s) \text{ and } 0 < s \\ f(x) & \text{if } \alpha^*(t) = (x, s) \text{ and } 0 > s \end{cases}$$

The path τ_i can be thought of as a projection of the piece $\alpha^*|_{(i,i+1)}$ to a cross-section of the vertex space between α_i and α_{i+1}. The annulus α is ρ-**thin** if $l(\tau_i) + 1 \leq \rho$ for all $i \in [-m, m-1]_{\mathbb{Z}}$. Here, the length is measured after tightening τ_i rel. end points to be locally injective. This is akin to putting a taxicab-like metric on the universal cover to measure distances in the vertex space. A family of annuli that will be useful in the next section are the 1-thin annuli. These are annuli whose trace of the basepoint projects to a null-homotopic path. One can also think of these annuli as homotopies of loops in the edge-space that respect the natural (semi)flow lines of M_f.

Now we can state the *combination theorem for mapping tori*:

Theorem 4.2 ([2, combination theorem]). *Let $f : \Gamma \to \Gamma$ be a π_1-injective map. If there are numbers $\lambda > 1$, $m \geq 1$, and a function $H : \mathbb{R} \to \mathbb{R}$ such that any ρ-thin annulus of length $2m$ with girth at least $H(\rho)$ is λ-hyperbolic, then $\pi_1(M_f)$ is word-hyperbolic.*

The hypothesis will be referred to as **the annuli flare condition**. Annuli α lift to hallways in the universal cover, $\tilde{\alpha} : I \times [-m, m] \to \tilde{M}_f$. In this context, the number ρ bounds the amount of shearing in the hallways and 1-thin means there is no shearing. H is a lower-bound for the thickness of the hallways' middle/girth. The flare condition is saying: Once we've bounded the shearing of the hallways, if we assume their girths are thick enough, then the hallways will get thicker towards one of their ends.

Technically, the combination theorem also requires that a *qi-embedded* condition is satisfied by the edge spaces, or equivalently, a quasi-convexity condition is satisfied by their fundamental groups. All our (vertex and edge) groups are finitely generated (f.g) free groups; f.g. subgroups of f.g. free groups are quasi-convex: free factors and finite index subgroups of f.g. free groups are quasi-convex and f.g. subgroups are free factors of finite index subgroups. So the condition is always satisfied under our assumptions.

Example 4.3 (Continuing to Example 3.5). We will exhibit two families of annuli: one that has 2^i-hyperbolic members for all $i \geq 1$; another that has no member that is λ-hyperbolic for some $\lambda > 1$. Let $c \subset G$ be the based loop defined earlier.

The first family: Fix $i \geq 1$, we have $g^{2i}(c) = c^{2^{2i}}$; therefore, the based loops c and $c^{2^{2i}}$ in the edge-space of M_g are homotopic in M_g. Let α be some homotopy between the two based loops that is also a 1-thin annulus of length $2i$. Thus, $l(\alpha_0) = l(g^i(c)) = l(c^{2^i}) = 2^i$, $l(\alpha_i) = 2^{2i}$, and $2^i l(\alpha_0) \leq l(\alpha_i)$. See the left half of Figure 2 for $i = 2$.

The second family: Fix $i \geq 2$ and let c' be the based loop given by $c'(x) = c(x + 1/2^i)$. Then $g^i(c') = c^{2^i} = g^i(c)$ as based loops; therefore, we get homotopies $c' \simeq g^{i-1}(c') \simeq c^{2^{i-1}} \simeq g(c)$ in M_g. Choose the homotopies so that their concatenation produces a 1-thin annulus of length $2i - 2$. Therefore, $l(\alpha_0) = l(g^{i-1}(c')) = 2^{i-1}$ but $l(c') = l(\alpha_{-i+1}) = 1$ and $l(\alpha_{i-1}) = l(g(c)) = l(c^2) = 2$. These annuli are the antithesis of being λ-hyperbolic. See the right half of Figure 2 for $i = 3$.
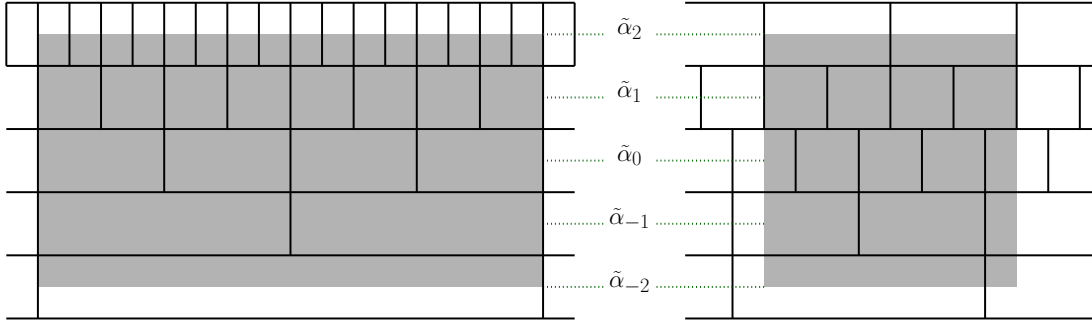

FIGURE 2. Lifts of two 1-thin annuli of length 4 to the universal cover $\tilde{M}_g$.

5. ANNULI AND PULLBACKS

Let $p : M_f \to S^1$ be given by $(x, t) \mapsto t$, α be an annulus of length $2m$, and $\beta : [-m, m] \to \mathbb{R}$, a lift of $p \circ \alpha^*$ to the universal cover of S^1.

Definition 5.1. An annulus α is **unidirectional** if β is either increasing or decreasing on $[-m, m]$, equivalently, $\beta|_{[-m,m]_{\mathbb{Z}}}$ is strictly increasing or decreasing. Otherwise, it is **bidirectional**.

Due to the fourth condition in the definition of an annulus, if α is bidirectional then β switches directions exactly once and it switches from increasing to decreasing.

Definition 5.2. An annulus α is **strictly bidirectional** if $\beta(-m) = \beta(m) - 1 < \beta(0)$.

We define a **homotopy of annuli** α, α' to be a homotopy that restricts to an edge space homotopy on each ring. Equivalently, a homotopy $\alpha \simeq \alpha'$ that induces a homotopy $\beta \simeq \beta'$ rel. integer points. By this definition, being (strictly) bidirectional becomes a property of the homotopy classes of annuli. For simplicity's sake, we choose class representatives that

are **tight annuli**, i.e., the rings are locally injective even at the basepoint. For the rest of this section, all annuli and loops are assumed to be tight.

In the language of annuli, pullback stabilizing can be interpreted as:

Proposition 5.3. *Suppose $f : \Gamma \to \Gamma$ is an immersion. Then $\hat{\Gamma}_L$ is empty for some L if and only if strictly bidirectional tight annuli have lengths uniformly bounded by some $2L$.*

Proof. There is a correspondence between (classes of) strictly bidirectional annuli in M_f and pullbacks of Γ. Let α be a strictly bidirectional annulus. For $i \in [-m, m]_\mathbb{Z}$, define:

$$\alpha_i' = \begin{cases} f(\alpha_i) & \text{if } -m \le i \le 0 \\ \alpha_i & \text{otherwise.} \end{cases}$$

The α_i' have the nice property that $f^{|i|}(\alpha_i') \simeq \alpha_0'$ in Γ. We can and will henceforth choose a homotopy class representative α so that $f(\alpha_i') = \alpha_{i+1}'$ if $i < 0$, $f(\alpha_i') = \alpha_{i-1}'$ if $i > 0$, and τ_i is null-homotopic rel. endpoints for all i. The last condition is equivalent to saying α is 1-thin.

Let $i > 0$. Since $f^i(\alpha_{-i}') = \alpha_0' = f^i(\alpha_i')$, by the definition of pullbacks, the pair $(\alpha_{-i}', \alpha_i')$ is a loop γ_i in Γ_i. Condition 4 in the definition of annuli and the assumption α is 1-thin imply $\gamma_1 \subset \hat{\Gamma}_1$, and more generally, $\gamma_i \subset \hat{\Gamma}_i$ for all i.

Conversely, a loop $(\sigma_-, \sigma_+) \subset \hat{\Gamma}_m$ completely determines a 1-thin strictly bidirectional annuli with length $2m - 2$; length can be extended to $2m$ if σ_- is the f-image of some loop in Γ. This in turn determines a homotopy class of annuli. Thus, we can view classes of strictly bidirectional annuli of length $2m$ as certain loops in $\hat{\Gamma}_m$. $\qquad\square$

Corollary 5.4. *Suppose $f : \Gamma \to \Gamma$ is an immersion and $\hat{\Gamma}_L = \emptyset$ for some L. Then, up to reversal of direction, an annulus α with length greater than $4L$ will satisfy*

$$\beta(-m) < \beta(0) < \beta(m) - 1.$$

Example 5.5 (Continuing Example 4.3)**.** The proof of Proposition 5.3 gives a correspondence between classes of strictly bidirectional annuli and components of $\hat{G}_i$. Since the latter are never empty, we get that M_g has strictly bidirectional annuli with unbounded lengths. Indeed, the second family of annuli we constructed in Example 4.3 are strictly bidirectional and unbounded; On the other hand, the first family are all unidirectional.

Example 5.6 (Continuing Example 3.6)**.** To contrast, we know that $\hat{H}_1$ consists of loops and $\hat{H}_2$ is empty. Since the definition of annulus requires them to have even length, M_h has no bidirectional annuli. If we relax the definition of annuli to allow odd length, then all bidirectional annuli in M_h have length 1 and are homotopies $a^k \simeq b^k$ of edge-space loops. Figure 3 is one explicit annulus with $k = 2$.

6. Hyperbolic Endomorphisms

An automorphism $\phi : G \to G$ of a word-hyperbolic group G is said to be **hyperbolic** if:

$\exists\, \lambda > 1,\, n \ge 1$ such that for all nontrivial $g \in G, \lambda|g| \le \max(|\phi^n(g)|, |\phi^{-n}(g)|)$
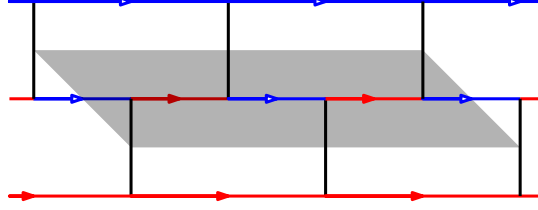
FIGURE 3. A lift of a 2-thin bidirectional annulus with length 1.

where $|\cdot|$ refers to the word-length with respect to some fixed finite generating set of G.

Bestvina-Feighn used their combination theorem to show that mapping tori of hyperbolic automorphisms are word-hyperbolic and we will prove an analogous statement for endomorphism of free groups (Theorem 6.4). First, we need to generalize the definition of hyperbolic automorphisms of free groups.

For the remainder of this section, assume a graph map $f : \Gamma \to \Gamma$ is a topological map that sends vertices to vertices and is locally injective on the interior of edges. In particular, f maps edges to nontrivial edge-paths but is not necessarily locally injective at the vertices. Let λ_f be the maximum $l(f(e))$ over all the edges e in Γ, where $l(\cdot)$ is the combinatorial lengths of edge-paths in Γ. By our assumptions, $\lambda_f \geq 1$.

Definition 6.1. For $\lambda > 1$ and $n \geq 1$, we say a graph map $f : \Gamma \to \Gamma$ is (λ, n)-**hyperbolic** if for all based nontrivial loops $\sigma : (S^1, *) \to \Gamma$, at least one of these two conditions holds:

(1) $\lambda l(f^n(\sigma)) \leq l(f^{2n}(\sigma))$
(2) $\lambda l(f^n(\sigma)) \leq l(\sigma)$

We will say (λ, n)-Condition 1/2 holds for σ if the corresponding condition above holds. Note that we assume the basepoint $\sigma(*)$ is a vertex of Γ, and the combinatorial lengths of based loops are measured after tightening rel. basepoint.

Remark 6.2. If $\lambda_f = 1$, then $l(f^n(\sigma)) = l(\sigma)$ for all $n \geq 1$ and f is not (λ, n)-hyperbolic since the definition of the latter requires $\lambda > 1$.

Lemma 6.3. *Suppose $f : \Gamma \to \Gamma$ is an immersion with no invariant loop of degree $d = 1$, i.e., f is atoroidal. Then f is $(2, n)$-hyperbolic for some $n \geq 1$.*

Proof. Suppose Γ has no nontrivial f-invariant forest. If there exists an edge e such that $l(f^n(e)) = 1$ for all $n \geq 1$. Then the f-orbit $\mathcal{O} = \cup_{n=1}^{\infty} f^n(e)$ is an f-invariant subgraph of Γ. By our assumption, $\mathcal{O}$ is not a forest, hence it contains an embedded loop σ. The orbit of σ consist of loops with the same length and there are only finitely many of them. Thus the orbit is eventually periodic, which contradicts the atoroidal assumption. Therefore, for all edges e, there is an $n_e \geq 1$ such that $l(f^{n_e}(e)) \geq 2$. Set $n = \max\{n_e \; : \; e$ is an edge.$\}$. Thus $l(f^n(e)) \geq 2$ for all edges e and f is $(2, n)$-hyperbolic since f is an immersion.

Suppose Γ has a nontrivial f-invariant forest. Collapse a maximal nontrivial f-invariant forest to produce a graph Γ' and map $f' : \Gamma' \to \Gamma'$. Since f is an immersion, it is a homeomorphism when restricted to a collapsed tree (component of forest). In particular,

distinct boundary points of a component of the forest have distinct f-images. Suppose $v' \in \Gamma'$ is the image of a vertex in Γ whose f-image is in a collapsed tree. The directions at v', $T_{v'}\Gamma'$, correspond to directions at the boundary of a collapsed tree $T_{\partial Y}\Gamma$ if v' itself is the image of a collapsed tree Y, otherwise they correspond to the directions of some vertex of Γ outside the collapsed forest F. Since f is injective on the boundary of a collapsed tree and df, the union of all derivative maps, is injective, then $df'_{v'}$ is also injective. Clearly, $df'_{v'}$ is also injective if v' is the image of a vertex in Γ whose f-image lies outside the collapsed forest. Therefore, df' is injective and f' is an immersion. By the previous paragraph, f' is $(2, n')$-hyperbolic for some n'. Choose k such that $2^k - 1 >$ number of edges in the maximal forest. If we let $n = kn'$, then for any immersed loop in Γ, any edge of the loop lying outside the maximal forest has f^n-image that is at least 2^k edges long, which will be longer than any immersed path lying in the forest. Thus f is $(2, n)$-hyperbolic. $\square$

We can now state a generalization of Bestvina-Feighn's theorem on mapping tori of hyperbolic automorphisms:

Theorem 6.4. *If $f : \Gamma \to \Gamma$ is a (λ, n)-hyperbolic graph map and strictly bidirectional annuli in M_f have lengths bounded by $2L$ for some $L > 1$, then $\pi_1(M_f)$ is word-hyperbolic.*

Remark 6.5. Gautero proved a weaker version of this theorem with the assumption that $f_*(\pi_1(\Gamma)) \leq \pi_1(\Gamma)$ is malnormal [8, Theorem 13.2]. This is equivalent to assuming M_f has no bidirectional annuli. It is worth noting that Gautero does not use the combination theorem.

Proof. Choose $k \geq 1$ so that $\lambda^k > 2$ and we have f is $(2, kn)$-hyperbolic. Fix $r \geq 1$ such that $2^r(2/\lambda_f^{kn})^L \geq 8$. Define $m = (2L + r)kn$ and

$$H(\rho) = \frac{4\rho}{\lambda_f - 1}(\lambda_f^{2m} - 1)2^m$$

By Remark 6.2, $\lambda_f > 1$ as f is (λ, n)-hyperbolic. Suppose α is a ρ-thin annulus of length $2m$ with $l(\alpha_0) \geq H(\rho)$. We will show that α is 7-hyperbolic. Without loss of generality, assume $\beta(-m) = \beta(0) - m$. The lower half of the annulus gives us the following homotopy in Γ rel. basepoint:

$$f^m(\alpha_{-m}) \simeq f^{m-1}(\tau_{-m}) \cdots \tau_{-1}\alpha_0\bar{\tau}_{-1} \cdots f^{m-1}(\bar{\tau}_{-m})$$

This homotopy allows us to bound the length of α_0 in terms of $f^m(\alpha_{-m})$:

$$l(f^m(\alpha_{-m})) \geq l(\alpha_0) - 2\sum_{i=0}^{m-1} l(f^{m-1-i}(\tau_{-m+i}))$$

$$\geq l(\alpha_0) - 2\sum_{i=0}^{m-1} \lambda_f^{m-1-i}\rho$$

$$= l(\alpha_0) - \frac{2\rho}{\lambda_f - 1}(\lambda_f^m - 1)$$

There are three cases to consider:

Case 1, $(2, kn)$-Condition 2 holds for $f^{m-kn}(\alpha_{-m})$.

Since $(2, kn)$-Condition 2 holds for $f^{m-kn}(\alpha_{-m})$, by induction, we get the inequality:

$$
\begin{aligned}
l(\alpha_{-m}) &\geq 2^{2L+r} l(f^m(\alpha_{-m})) \\
&\geq 2^{2L+r} l(\alpha_0) - \frac{2\rho}{\lambda_f - 1}(\lambda_f^m - 1) 2^{2L+r} \\
&> 8l(\alpha_0) - H(\rho) \\
&\geq 7l(\alpha_0) \qquad\qquad\qquad\qquad\qquad\qquad \text{as } l(\alpha_0) \geq H(\rho)
\end{aligned}
$$

Case 2, α is unidirectional and $(2, kn)$-Condition 1 holds for $f^{m-kn}(\alpha_{-m})$.

The top half of the annulus gives us:

$$
f^{2m}(\alpha_{-m}) \simeq f^{2m-1}(\tau_{-m}) \cdots \tau_{m-1} \alpha_m \bar{\tau}_{m-1} \cdots f^{2m-1}(\bar{\tau}_{-m})
$$

Since $(2, kn)$-Condition 1 holds for $f^{m-kn}(\alpha_{-m})$, by induction, we get the inequality:

$$
\begin{aligned}
2^{2L+r}\left(l(\alpha_0) - \frac{2\rho}{\lambda_f - 1}(\lambda_f^m - 1)\right) &\leq 2^{2L+r} l(f^m(\alpha_{-m})) \\
&\leq l(f^{2m}(\alpha_{-m})) \\
&\leq l(\alpha_m) + 2\sum_{i=0}^{2m-1} l(f^{m-1-i}(\tau_{-m+i})) \\
&\leq l(\alpha_m) + \frac{2\rho}{\lambda_f - 1}(\lambda_f^{2m} - 1)
\end{aligned}
$$

Rearranging this inequality:

$$
\begin{aligned}
l(\alpha_m) &\geq 2^{2L+r} l(\alpha_0) - \frac{2\rho}{\lambda_f - 1}(\lambda_f^m - 1) 2^{2L+r} - \frac{2\rho}{\lambda_f - 1}(\lambda_f^{2m} - 1) \\
&> 8l(\alpha_0) - H(\rho) \\
&\geq 7l(\alpha_0)
\end{aligned}
$$

Case 3, α is bidirectional and $(2, kn)$-Condition 1 holds for $f^{m-kn}(\alpha_{-m})$.

By the hypothesis on strictly bidirectional annuli and inequality $m > 2L$, there exists $s, t \geq 1$ such that $\beta(0) < \beta(s) = \beta(m) - 1 < \beta(s + t) = \beta(s + t + 1)$. The top half of the annulus induces two homotopies:

$$
f^{m+s+t}(\alpha_{-m}) \simeq f^{m+s+t-1}(\tau_{-m}) \cdots \tau_{s+t-1} \alpha_{s+t} \bar{\tau}_{s+t-1} \cdots f^{m+s+t-1}(\bar{\tau}_{-m})
$$

$$
f^t(\alpha_m) \simeq f^{t-1}(\bar{\tau}_{m-1}) \cdots \bar{\tau}_{s+t} f(\alpha_{s+t}) \tau_{s+t} \cdots f^{t-1}(\tau_{m-1})
$$

Choose $u \in [0, kn)_{\mathbb{Z}}$ so that $v = \frac{s+t+u}{kn} = \left\lceil \frac{s+t}{kn} \right\rceil$ is an integer. Since $(2, kn)$-Condition 1 holds for $f^{m-kn}(\alpha_{-m})$, we get

$$
\text{(6.1)} \qquad\qquad 2^v l(f^m(\alpha_{-m})) \leq l(f^u(\alpha_{s+t})) + \frac{2\rho}{\lambda_f - 1}(\lambda_f^{2m} - 1)\lambda_f^u
$$

The second homotopy gives us:

$$(6.2) \qquad \lambda_f^{t+u-1} l(\alpha_m) \geq l(f^{t+u-1}(\alpha_m)) \geq l(f^u(\alpha_{s+t})) - \frac{2\rho}{\lambda_f - 1}(\lambda_f^t - 1)\lambda_f^{u-1}$$

Rewrite (6.2) and combine it with (6.1) to get

$$2^v l(f^m(\alpha_{-m})) \leq \lambda_f^{t+u-1} l(\alpha_m) + \frac{2\rho}{\lambda_f - 1}(\lambda_f^t - 1)\lambda_f^{u-1} + \frac{2\rho}{\lambda_f - 1}(\lambda_f^{2m} - 1)\lambda_f^u$$

$$\therefore \quad \frac{2^v}{\lambda_f^{t+u-1}}\left(l(\alpha_0) - \frac{2\rho}{\lambda_f - 1}(\lambda_f^m - 1)\right) \leq \frac{2^v}{\lambda_f^{t+u-1}}l(f^m(\alpha_{-m}))$$

$$\leq l(\alpha_m) + \frac{2\rho}{\lambda_f - 1}\frac{\lambda_f^t - 1}{\lambda_f^t} + \frac{2\rho}{\lambda_f - 1}\frac{\lambda_f^{2m} - 1}{\lambda_f^{t-1}}$$

$$\leq l(\alpha_m) + \frac{4\rho}{\lambda_f - 1}(\lambda_f^{2m} - 1) \qquad \text{since } \lambda_f \geq 1$$

By construction and hypothesis, $m = (2L + r)kn = s + 2t$, $u < kn$, and $t < L$. These can be used to show $v > L + r$ and $t + u - 1 \leq L(kn)$, so $\frac{2^v}{\lambda_f^{t+u-1}} > 2^r \left(\frac{2}{\lambda_f^{kn}}\right)^L \geq 8$ from our choice of r. Therefore,

$$l(\alpha_m) \geq 8l(\alpha_0) - \frac{2\rho}{\lambda_f - 1}(\lambda_f^m - 1)8 - \frac{4\rho}{\lambda_f - 1}(\lambda_f^{2m} - 1)$$

$$> 8l(\alpha_0) - H$$

$$\geq 7l(\alpha_0)$$

We have covered all the cases and shown that M_f satisfies the annuli flare condition. By the combination theorem, $\pi_1(M_f)$ is word-hyperbolic. $\qquad \square$

Example 6.6 (Continuing Example 3.6). Since $\hat{H}_2$ is empty, the Sapir group is word-hyperbolic.

Theorem 6.7. *Suppose $f : \Gamma \to \Gamma$ is an immersion. The following are equivalent:*

(1) $\pi_1(M_f)$ *is word-hyperbolic.*
(2) $\pi_1(M_f)$ *contains no $BS(1, d)$ subgroups for $d \geq 1$.*
(3) f *has no invariant loop.*
(4) f *is (λ, n)-hyperbolic and $\hat{\Gamma}_L = \emptyset$ for some $L \geq 1$.*

Proof.

(1) $\Longrightarrow$ (2): $BS(1, d)$ subgroups are an obstruction to hyperbolicity.

(2) $\Longrightarrow$ (3): Due to Kapovich [10, Lemma 2.3]:

It is easy to see that having an f-invariant loop with degree d implies there is a homomorphism $BS(1, d) \to \pi_1(M_f)$. Using normal forms, one shows that the homomorphism is injective.

(3) $\Longrightarrow$ (4): This is Lemma 6.3 and Proposition 3.11.

(4) $\Longrightarrow$ (1): This is Proposition 5.3 and Proposition 6.4. $\qquad \square$

To remove the immersion assumption, it suffices to show that for any π_1-injective map $f : \Gamma \to \Gamma$ with no invariant loop, the following holds:

(1) f is (λ, n)-hyperbolic — this is analogous to Brinkmann's theorem/Lemma 6.3.
(2) pullbacks stabilize — this is analogous to Proposition 3.11.

Combining Theorem 6.7 with Remark 3.12, we get:

Corollary 6.8. *For any immersion f, there is a computable integer $L(f)$ such that f has no invariant loop with degree $d \geq 2$ if and only if $\hat{\Gamma}_{L(f)}$ is empty.*

7. Fully Irreducible Endomorphisms

In this section, F_n is a free group with finite rank $n \geq 2$.

Definition 7.1. An injective endomorphism $\phi : F_n \to F_n$ is **fully irreducible** or **irreducible with irreducible powers (iwip)** if there are no positive integer k, proper free factor $A \leq F_n$, and element $x \in F_n$ such that $\phi^k(A) \leq xAx^{-1}$.

The following is an unpublished theorem due to Patrick Reynolds. Reynolds' statement is stronger than the version given here but this weaker form suffices for our purposes.

Theorem 7.2 ([14, Corollary 5.5]). *If $\phi : F_n \to F_n$ is nonsurjective and fully irreducible, then it is induced by an immersion $f : \Gamma \to \Gamma$.*

Corollary 7.3. *Let $\phi : F_n \to F_n$ be a fully irreducible endomorphism. The following are equivalent:*

(1) F_n*_ϕ *is word-hyperbolic.*
(2) F_n*_ϕ *has no $BS(1, d)$ subgroups for $d \geq 1$.*
(3) *There are no $k, d \geq 1$ and nontrivial $g \in F_n$ such that $[\phi^k(g)] = [g^d]$.*

Proof. We need to show (3) $\implies$ (1): If ϕ is surjective, then this is Brinkmann's theorem. So we may assume ϕ is nonsurjective. By Reynolds' theorem, ϕ is induced by an immersion $f : \Gamma \to \Gamma$. The implication now follows from Theorem 6.7. $\qquad\square$

The following related corollary was suggested by Ilya Kapovich.

Corollary 7.4. *Let $\phi : F_2 \to F_2$ be an injective endomorphism. The following are equivalent:*

(1) F_2*_ϕ *is word-hyperbolic.*
(2) F_2*_ϕ *has no $BS(1, d)$ subgroups for $d \geq 1$.*
(3) *There are no $k, d \geq 1$ and nontrivial $g \in F_n$ such that $[\phi^k(g)] = [g^d]$.*

Proof. We need to show (3) $\implies$ (1): In $F_2 = \langle a, b \rangle$, automorphisms either fix the conjugacy class of $[a, b] = aba^{-1}b^{-1}$ or map it to the class of its inverse $bab^{-1}a^{-1}$. Since ϕ is atoroidal, it must be nonsurjective. Proper free factors of F_2 are infinite cyclic; as no power of ϕ maps a nontrivial element to a conjugate of its power, then no power of ϕ maps a proper free factor into a conjugate of itself. Thus ϕ is fully irreducible. By Reynolds' theorem, ϕ is induced by an immersion $f : \Gamma \to \Gamma$. The lack of f-invariant loops implies $\pi_1(M_f) \cong F_2*_\phi$ is word-hyperbolic by Theorem 6.7. $\qquad\square$

We will now give some examples. For $n \in \{2, 3, 4\}$, let $\varphi_n : F_n \to F_n$ be defined by:

$$\varphi_2 : \begin{array}{l} a \mapsto ba^{-1} \\ b \mapsto a^2(ab^{-1})^2 a^2 \end{array} \qquad \varphi_3 : \begin{array}{l} a \mapsto b \\ b \mapsto aca^{-1} \\ c \mapsto ab^{-1}ac^{-1}a^{-1}ba \end{array} \qquad \varphi_4 : \begin{array}{l} a \mapsto b \\ b \mapsto ac \\ c \mapsto da^{-1} \\ d \mapsto ab^{-1}ad^{-1}b^{-1}aca \end{array}$$

$G = F_2*_{\varphi_2}$ is a 1-relator group and we used Brown's algorithm [5] to compute the BNS invariant, a cone $\Sigma(G) \subset H^1(G; \mathbb{R})$. The two other endomorphisms above were found by choosing two rational rays in $\Sigma(G)$ and rewriting the presentation of G as an asc. HNN-ext. with respect to these rays. So $G \cong F_3*_{\varphi_3} \cong F_4*_{\varphi_4}$. This group's BNS invariant has no symmetric subset, so we can conclude that G never splits as a free-by-cyclic group [3]. Consequently, Brinkmann's result (and Kapovich's as far as we can tell) can not be used to prove word-hyperbolicity of G.

Claim. *No iterate of φ_2, φ_3, or φ_4 maps a nontrivial element to a conjugate of its power and G is word-hyperbolic.*

Sketch proof. By the preceding discussion, these endomorphisms were constructed so that their HNN-extensions are isomorphic to the same 1-relator group G. The endomorphism φ_3 is induced by an *irreducible* immersion f_3 (See second graph in Figure 4). This immersion has *Perron-Frobenius* eigenvalue $\lambda_3 = 1 + \sqrt{3}$ which is not a root of any integer. As, f_3 expands all immersed loops uniformly by λ_3, it has no invariant loop. By Corollary 7.3, $G \cong F_3*_{\varphi_3}$ is word-hyperbolic. But $G = F_2*_{\varphi_2} \cong F_4*_{\varphi_4}$. Applying Corollary 7.3 again, no iterate of φ_2 and φ_4 maps a nontrivial element to a conjugate of its power. $\square$

As argued in the proof of Corollary 7.4, in F_2, the lack of φ_2^k-invariant cyclic subgroups, for all $k \geq 1$ implies the endomorphism φ_2 is fully irreducible and induced by an immersion. We found immersions inducing both φ_2 and φ_4 (Figure 4).

An interesting area of further study is characterizing all the nonsurjective injective endomorphisms that are not covered by the main theorem; more precisely, which nonsurjective endomorphisms are not induced by an immersion. For F_2, the answer seems to be only endomorphisms that take the following form up to change of basis:

$$\psi : \begin{array}{l} a \mapsto a \\ b \mapsto a^k b \ldots b \end{array} \qquad \text{or} \qquad \psi : \begin{array}{l} a \mapsto a^{-1} \\ b \mapsto (a^k b \ldots b)^{-1} \end{array} \qquad (k \geq 1)$$
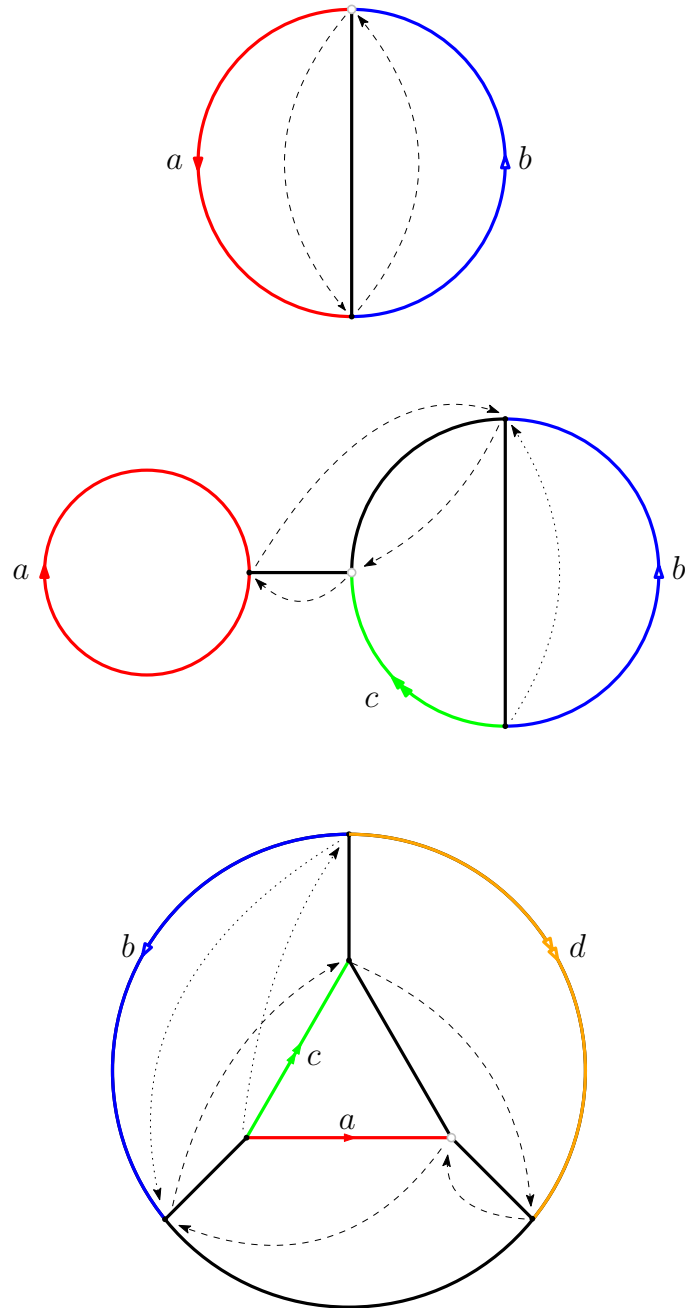
FIGURE 4. Markings for the graph immersions inducing φ_i.
The dashed/dotted arrows shows the orbits of the vertices under the immersions. The white vertex is the marked point. The marking, vertex orbits, and endomorphisms' definitions are enough to (re)construct the immersions.

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