Limit trees for free group automorphisms: universality

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Abstract

To any free group automorphism, we associate a universal (cone of) limit tree(s) with three defining properties: first, the tree has a minimal isometric action of the free group with trivial arc stabilizers; second, there is a unique expanding dilation of the tree that is equivariant with respect to the automorphism; and finally, the loxodromic elements are exactly the elements that weakly limit to topmost attracting laminations under forward iteration by the automorphism. So the action on the tree detects the automorphism's topmost exponential dynamics.

As a corollary, our previously constructed limit pretree that detect the exponential dynamics is canonical; the pretree admits pseudometrics that can be viewed as a universal hierarchy of limit trees. For atoroidal automorphisms, this universal hierarchical decomposition is analogous to the Nielsen–Thurston normal form for a surface homeomorphism or the Jordan canonical form for a linear map. We use it to sketch a proof that atoroidal outer automorphisms have virtually abelian centralizers.

Introduction

We previously constructed a "limit pretree" that detects the exponential dynamics for an arbitrary free group automorphism [Mut21b]. In this sequel, we prove the construction is canonical. This completes the existence and uniqueness theorem for a free group automorphism's limit pretree. In [Mut21b], we motivated the existence and uniqueness theorem by describing it as a free group analogue to the Nielsen–Thurston theory for surface homeomorphisms, which in turn can be seen as the surface analogue to the Jordan canonical form for linear maps. We now give our own motivation for this result.

Universal representation of an endomorphism

It feels rather odd to discuss my personal motivation while using the communal "we"; excuse me as I break this convection a bit for this section. In my doctoral thesis, I extended

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Brinkmann's hyperbolization theorem to mapping tori of free group endomorphisms. This required studying the dynamics of endomorphisms. Along the way, I proved that injective endomorphisms have canonical representatives. More precisely, suppose $\phi\colon F \to F$ is an injective endomorphism of a finitely generated free group; then there is:

1. a minimal simplicial F-action on a simplicial tree T with trivial edge stabilizers;
2. a ϕ-*equivariant* expanding embedding $f\colon T \to T$ (unique up to isotopy); and
3. an element in F is T-elliptic if and only if one of its forward ϕ-iterate is conjugate to an element in a $[\phi]$-*periodic* free factor of F.

Existence of the "limit free splitting" (i.e. T with its F-action) for the *outer class* $[\phi]$ was the core of my thesis (see also [Mut21a, Theorem 3.4.5]). Universality follows from *bounded cancellation*: any other simplicial tree T' satisfying these three condition will be uniquely equivariantly isomorphic to T [Mut21a, Proposition 3.4.6].

In a way, the limit free splitting detects and filters the "nonsurjective" dynamics of the (outer) endomorphism. When $\phi\colon F \to F$ is an automorphism, then T is a singleton and the free splitting provides no new information. On the other extreme, the F-action on T can be free; in this case, let $\Gamma \coloneqq F\backslash T$ be the quotient graph. Then the outer endomorphism $[\phi]$ is represented by a unique expanding *immersion* $[f]\colon \Gamma \to \Gamma$ and $[\phi]$ is *expansive* — such outer endomorphisms are characterized by the absence of $[\phi]$-periodic (conjugacy classes of) nontrivial free factors [Mut21a, Corollary 3.4.8]. The most important thing is that the expanding immersion $[f]$ has nice dynamics and greatly simplifies the study of expansive outer endomorphisms.

After completing my thesis, I found myself in a paradoxical situation: I had a better "understanding" of nonsurjective endomorphisms than automorphisms — the main obstacle to studying the dynamics of nonsurjective endomorphisms was understanding the dynamics of automorphisms. The naïve expectation (when I started my thesis) had been that a nonsurjective endomorphism has more complicated dynamics as it is not invertible. The current project was born out of an obligation to correct this imbalance.

Universal representation of an automorphism

What follows is a direct analogue of the above discussion in the setting of automorphisms. The existence theorem [Mut21b, Theorem III.3] produces an action that detects and filters the "exponential" dynamics of an automorphism. Specifically, suppose $\phi\colon F \to F$ is an automorphism of a finitely generated free group. Then there is:

1. a minimal *rigid* F-action on a *real pretree* T with trivial arc stabilizers;
2. a unique ϕ-equivariant "F-expanding" pretree-automorphism $f\colon T \to T$; and
3. an element in F is T-elliptic if and only if it *grows polynomially with respect to* $[\phi]$.

The pair of the pretree T and its rigid F-action is called a *(forward) limit pretree* for the outer automorphism $[\phi]$. The theorem is stated properly in Chapter III as Theorem III.1. When $[\phi]$ is *polynomially growing*, then *the* limit pretree is a singleton (and hence unique) but provides no new information. We are mainly interested in *exponentially growing* $[\phi]$ as their limit pretrees are not singletons. On the other hand, the F-action on a limit pretree is free if and only if $[\phi]$ is *atoroidal*, i.e. there are no $[\phi]$-periodic (conjugacy classes of) nontrivial elements [Mut21b, Corollary III.4]. As with expanding immersions and expansive outer endomorphisms, the unique expanding "homeomorphism" $[f]$ (on the quotient space $F\backslash T$) has dynamics that could facilitate the study of atoroidal outer automorphisms.

Unlike the endomorphism case, uniqueness of limit pretrees requires a more involved argument. It was remarked in the epilogue of [Mut21b] that the only source of indeterminacy in the existence proof was [Mut21b, Proposition III.2]; this proposition is restated in Section I.4 as Proposition I.2 and a proof is sketched in Sections II.1 and II.4. The main result of this paper is a universal version of the proposition. It can also be thought of as an existence and uniqueness theorem for an action that detects and filters the "topmost" exponential dynamics of an outer automorphism:

Main Theorem (Theorems III.7 and IV.2)**.**
Let $\phi\colon F \to F$ be an exponentially growing automorphism of a finitely generated free group and $(\mathcal{A}_j^{top}[\phi])_{j=1}^k$ the $[\phi]$-orbits of its topmost attracting laminations*.*
Then there is:

1. *a minimal* factored F-tree $(Y, \oplus_{j=1}^k \delta_j)$ *with trivial arc stabilizers;*
2. *a unique ϕ-equivariant* expanding dilation $h\colon (Y, \oplus_{j=1}^k \delta_j) \to (Y, \oplus_{j=1}^k \delta_j)$*; and*
3. *for $1 \le j \le k$, a nontrivial element in F is* δ_j-loxodromic *if and only if its forward ϕ-iterates have axes that* weakly limit *to $\mathcal{A}_j^{top}[\phi]$;*

moreover, the factored F-tree $(Y, \oplus_{i=1}^k \delta_i)$ is unique up to a unique equivariant dilation.

Thus the factored tree is a universal construction for the exponentially growing outer automorphisms and we call it *the topmost limit tree*. For completeness, define the topmost limit tree for polynomially growing $[\phi]$ to be a/the singleton. As an immediate corollary, the previously constructed limit pretrees are independent of the choices made in the proof of Theorem III.1, i.e. *the* limit pretree is canonical (Corollary III.9). Let us now briefly define the emphasized terms in the theorem's statement.

An *F-tree* is an ($\mathbb{R}$-)tree with an isometric F-action. Informally, an F-tree is *factored* if its metric has been equivariantly decomposed as a sum $\oplus_{j=1}^k \delta_j$ of "length measures". For a factored F-tree $(Y, \oplus_{j=1}^k \delta_j)$, an element in F is *δ_i-loxodromic* if it has an axis in Y with "positive δ_i-measure". An equivariant homeomorphism $(T, \oplus_{j=1}^k d_j) \to (Y, \oplus_{j=1}^k \delta_j)$ of factored F-trees is a *dilation* if it is a homothety of each pair of *factors* d_j and δ_j. A dilation is *expanding* if each factor-homothety is expanding.

A *lamination* in F is a nonempty closed subset in the *space of lines* in F. A sequence of lines (e.g. axes) *weakly limits* to a lamination if some subsequence converges to the lamination. Any exponentially growing $[\phi]$ has a nonempty partially ordered finite set of *attracting laminations* with an order-preserving $[\phi]$-action; the maximal elements of the partial order are called *topmost*.

Some applications of universal representations

Fix a finitely generated free group F and an automorphism $\phi\colon F \to F$. Since the (equivariant dilation class of a) topmost limit tree $(Y, \oplus_{j=1}^{k} \delta_j)$ for $[\phi]$ is well-defined, any equivariant dilation invariant of the tree is automatically an invariant of $[\phi]$.

For instance, Gaboriau–Levitt index $i(Y)$ (as defined in [GL95, Chapter III]) of the topmost limit tree for $[\phi]$ is the *topmost (forward) index for* $[\phi]$. In fact, since the limit pretree T for $[\phi]$ is canonical, its index $i(T)$ (as defined in [Mut21b, Appendix A]) is the *exponential (forward) index for* $[\phi]$; when $[\phi]$ is atoroidal, the index $i(T)$ is related to the Gaboriau–Jaeger–Levitt–Lustig index for $[\phi]$ defined in [GJLL98, Section 6]. Each factor δ_j induces a metric on an associated minimal F-tree (Y_j^{top}, δ_j); the pairing of δ_j and topmost attracting lamination $\mathcal{A}_j^{top}[\phi]$ means $i(Y_j^{top})$ is an *index for* $\mathcal{A}_j^{top}[\phi]$. Finally, structural decompositions of Y or T give canonical $[\phi]$-invariant decompositions of the free group.

A generalization of the main theorem: An attracting lamination $\mathcal{A}[\phi]$ for $[\phi]$ with *stretch factor* λ is dominating if any distinct attracting lamination $\mathcal{A}'[\phi]$ containing $\mathcal{A}[\phi]$ has a strictly smaller *stretch factor* $\lambda' < \lambda$. Topmost attracting laminations are vacuously dominating; moreover, the $[\phi]$-action permutes the dominating attracting laminations. Similar techniques should imply the main theorem still holds with respect to $[\phi]$-orbits of dominating attracting laminations $\mathcal{A}_j^{dom}[\phi]$ — the factored tree is *the dominating limit tree* for $[\phi]$. This generalization would determine all trees that support a ϕ-equivariant expanding homothety! A proof of this will appear elsewhere; for now, we suggest one use-case:

Corollary (conditional)**.** *If $\phi\colon F \to F$ is an atoroidal automorphism, then the centralizer of $[\phi]$ in the outer automorphism group* $\mathrm{Out}(F)$ *is virtually a free abelian group with rank at most the number of $[\phi]$-orbits of attracting laminations for $[\phi]$.*

Proof outline, assuming the universal construction of dominating limit trees.

Let $(Y, \oplus_{j=1}^{k_1} \delta_j)$ be the topmost limit tree for $[\phi]$ and $C[\phi]$ the centralizer for $[\phi]$ in $\mathrm{Out}(F)$. Replace $C[\phi]$ with a finite index subgroup if necessary and assume it acts trivially on the attracting laminations for $[\phi]$. If $[\phi'] \in C[\phi]$, then each minimal F-tree (Y_j^{top}, δ_j) for $[\phi]$ supports a ϕ'-equivariant λ_j'-homothety by uniqueness of the topmost limit tree for $[\phi]$. Thus we can define a group homomorphism $\ell^{top}\colon C[\phi] \to \mathbb{R}^{k_1}$ that maps $[\phi']$ to $(\log(\lambda_j'))_{j=1}^{k_1}$. If $\lambda_j' > 1$, then uniqueness of the dominating limit tree for $[\phi']$ implies (Y_j^{top}, δ_j) is the associated F-tree for some factor in the dominating limit tree for $[\phi']$; in particular, λ_j' is an exponential $[\phi']$-growth rate for some element in F. So the image of $C[\phi]$ under each coordinate projection ℓ_j^{top} of ℓ^{top} is a discrete subgroup of $\mathbb{R}$ — the nonzero images are bounded away from 0 by a constant that depends only on $\mathrm{rank}(F)$.

The subgroup system $\mathcal{G}_2 \coloneqq \mathcal{G}[Y]$ of point stabilizers in the topmost limit tree for $[\phi]$ is also $[\phi']$-invariant. The restriction of $[\phi']$ to $\mathcal{G}_2$ will commute with the restriction $[\phi_2]$ of $[\phi]$ to $\mathcal{G}_2$. As above, we define a group homomorphism $\ell^{(2)} \colon C[\phi] \to \mathbb{Z}^{k_2}$. Continue this inductive process until the action on $Y^{(n)}$ is free — this will happen as $[\phi]$ is atoroidal.

This process constructs a group homomorphism $\ell \colon C[\phi] \to \mathbb{Z}^k$, where $k = \sum_{i=1}^n k_i$. Replace $C[\phi]$ with a finite index subgroup if necessary and assume it acts trivially on the orbits of branch points of T and directions at branch points. As the F-action on the limit pretree T is free, the kernel $\ker(\ell)$ is trivial — see Kapovich–Lustig's Propositions 4.1–4.2 in [KL11]. So $C[\phi]$ is free abelian. □

Bestvina–Feighn–Handel previously proved that centralizers of *fully irreducible* outer automorphisms are virtually cyclic [BFH97, Theorem 2.14]. Prior to this corollary, we did not even know whether the centralizers of atoroidal outer automorphisms were finitely generated. To get a complete characterization of arbitrary centralizers, we need nondegenerate universal representations for polynomially growing automorphisms. We still do not know whether arbitrary centralizers are finitely generated.

Some historical context. This paper continues Gaboriau–Levitt–Lustig's philosophy of using limit trees to give an alternative proof of the Scott conjecture [GLL98]. In particular, our paper relies only on the existence of irreducible train tracks [BH92, Section 1] but none of the typical *splitting paths* analysis. Zlil Sela gave another dendrogical proof the conjecture (Bestvina–Handel's theorem) that used the *Rips' theorem* in place of train track technology [Sel96]. Frédéric Paulin gave yet another dendrological proof that avoids both train tracks and Rips' theorem [Pau97].

About the same time, Bestvina–Fieghn–Handel then used train tracks and trees to prove fully irreducible (outer) automorphisms have universal limit trees [BFH97]. They used this to give a short dendrological proof of a special case of the Tits Alternative for $\mathrm{Out}(F)$; their later proof of the general case was much more technical due to the lack of similar universal limit trees [BFH00]. Universal limit trees have been indispensable for studying fully irreducible automorphisms. In principle, a universal construction of limit trees for all automorphisms would lead to a dendrological proof of the Tits Alternative and extend much of the theory for fully irreducible automorphisms to arbitrary automorphisms.

In a sequel to [Sel96], Sela used limit trees and Rips' theorem to give a canonical hierarchical decomposition of the free group F that is invariant under a given atoroidal automorphism [Sel95]. This second paper was never published and a third announced paper that extends the canonical decomposition to arbitrary automorphisms was never released even as a preprint (as far as we know). We remark that the limit trees used in that paper were not (or rather, were never proven to be) canonical/universal. Perhaps, one could combine the canonical decomposition with Bestvina–Feighn–Handel dendrological proof to give a universal construction of limit trees for atoroidal automorphisms — our approach is independent of Sela's work and applies more generally to exponentially growing automorphisms. Conversely, we suspect that a careful study of the structure of our universal topmost limit trees might recover Sela's canonical hierarchical decomposition.

Morgan–Shalen introduced the term "$\mathbb{R}$-trees" in [MS84]. They also defined "Λ-trees" for an ordered abelian group Λ, say a subgroup of $\mathbb{R}^n$ with lexicographic ordering. At first glance, the *hierarchy of pseudometrics* on a real pretree (see Section I.2) might look like a Λ-tree. But in our constructed hierarchies, paths "exit" *infinitesimal* trees through metric completion points; whereas, in a Λ-tree, paths exit through points at infinity.

Proof outline for the existence theorem (Theorem III.7)

We now outline our proof of the existence theorem. One method for constructing limit trees is iterating *expanding irreducible train tracks*, which can be traced back as far as Bestvina–Feighn's preprint [BF94]. This carried out in Section II.1 but it has to two drawbacks: exponentially growing automorphisms do not always have expanding irreducible train tracks; and even when they do, the point stabilizers of the corresponding limit tree are not canonical as they can change with the choice of train tracks. We handle the first obstacle in Section II.4 by constructing a limit tree (Y_1, δ_1) using a descending sequence of irreducible train tracks, where only the last train track is expanding. Such descending sequences always exist for exponentially growing automorphisms.

Next, we construct in Section III.1 a pretree with an F-action whose point stabilizers are canonical. Set $G_1 \coloneqq F$ and let $\mathcal{G}_2$ be the $[\phi]$-invariant subgroup system determined by the point stabilizers of G_1 acting on Y_1. By restricting $[\phi]$ to $\mathcal{G}_2$ and inductively repeating the construction, we get a descending sequence of limit forests $(\mathcal{Y}_i, \delta_i)_{i=1}^n$. Each limit forest $(\mathcal{Y}_i, \delta_i)$ has attracting laminations $\mathcal{A}_i[\phi]$ for $[\phi]$ that is the forward limit of any $\mathcal{Y}_i$-loxodromic element in $\mathcal{G}_i$. Starting with $X^{(1)} = Y_1$, replace the points in $X^{(i-1)}$ fixed by $\mathcal{G}_i$ with the pretrees $\mathcal{Y}_i$ to produce $X^{(i)}$ for $1 < i \leq n$. The limit pretree $T = X^{(n)}$ has canonical point stabilizers: the unique maximal polynomially growing subgroups. Everything we have mentioned so far is a rehash of [Mut21b]. From the blow-up construction, the limit pretree T inherits an F-invariant hierarchy $(\delta_i)_{i=1}^k$ of pseudometrics — the pseudometric δ_i is defined on maximal $\mathcal{G}_i$-invariant convex subsets of T that have 0 δ_{i-1}-diameter.

The theorem is finally proven in Section III.4. The key insight for the proof: if attracting laminations $\mathcal{A}_i[\phi]$ are topmost, then the $\mathcal{G}_i$-invariant pseudometric δ_i can be extended to an F-invariant pseudometric on T. Let $(\mathcal{A}_{\iota(j)}[\phi])_{j=1}^k$ be the subsequence of topmost attracting laminations and $\delta_{\iota(j)}$ the corresponding F-invariant pseudometrics on T. Then their sum, denoted $\oplus_{j=1}^k \delta_{\iota(j)}$, is an F-invariant *factored* pseudometric on T whose associated metric space is a topmost limit forest. This concludes the outline.

In Section III.5, we show that the topmost limit tree is independent of the choices made in the construction; in Section IV.2, we prove universality. These proofs rely on Chapter V: three variations of Bestvina–Feighn–Handel's convergence criterion [BFH97, Lemma 3.4].

We use the results of [Mut21b] as black boxes and the two papers can be read in any order.

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Contents

I Preliminaries

In this paper, F will denote a free group with $2 \leq \text{rank}(F) < \infty$. Subscripts will never indicate the rank but will instead be used as indices. For inductive arguments, we also work with a free group system of finite type: disjoint union $\mathcal{F} = \bigsqcup_{j \in J} F_j$ of nontrivial finitely generated free groups F_j indexed by a possibly empty finite set J. In this paper, $\mathcal{F}$ is always a free group system of finite type with some component F_j that is not cyclic.

I.1 Group systems and actions

Nearly all statements and results about groups and connected spaces that we are interested in still hold when "connectivity" is relaxed and we work with "systems" componentwise. In general (almost categorical) terms, a system of [?-objects] is a disjoint union $\mathcal{O} = \bigsqcup_{j \in J} O_j$ of [?-objects] O_j indexed by some set J. An [?-isomorphism] of systems $\psi\colon \mathcal{O} \to \mathcal{O}'$ is a bijection $\sigma\colon J \to J'$ of the corresponding indexing sets and a union of [?-isomorphisms] $\psi_j\colon O_j \to O'_{\sigma \cdot j}$. The calligraphic font is reserved for systems.

In more concrete terms, here are some basic concepts that will show up in the paper:

1. an isomorphism of group systems $\psi\colon \mathcal{G} \to \mathcal{G}'$ is a bijection whose restriction to any component $G_j \subset \mathcal{G}$ is a group isomorphism of components; for group systems, we always assume (for convenience) components are nontrivial if the system is nonempty.

2. two isomorphisms of group systems $\psi, \psi'\colon \mathcal{G} \to \mathcal{G}'$ are in the same outer class $[\psi]$ if the component isomorphisms $\psi_j, \psi'_j\colon G_j \to G'_{\sigma \cdot j}$ differ only by post-composition with an inner automorphism of $G'_{\sigma \cdot j}$ for all $j \in J$.

3. a metric on a set system $\mathcal{X}$ is a disjoint union of metrics $d_j\colon X_j \times \mathcal{X}_j \to \mathbb{R}_{\geq 0}$ on the components $X_j \subset \mathcal{X}$.

4. for a group system $\mathcal{G}$, a $\mathcal{G}$-action on an object system $\mathcal{O}$ (or $\mathcal{G}$-object system $\mathcal{O}$) is a union of component G_j-actions on $O_{\beta \cdot j}$ (or G_j-objects $O_{\beta \cdot j}$) for some bijection $\beta\colon J \to J'$.

5. for an automorphism of group system $\psi\colon \mathcal{G} \to \mathcal{G}$ and a $\mathcal{G}$-object system $\mathcal{O}$, the ψ-twisted $\mathcal{G}$-object system $\mathcal{O}\psi$ is given by precomposing the component $G_{\sigma \cdot j}$-action on $O_{\beta\sigma \cdot j}$ with the component isomorphism $\psi_j\colon G_j \to G_{\sigma \cdot j}$ to get a G_j-object $O_{\beta\sigma \cdot j}$.

I.2 Pretrees, trees, and hierarchies

Pretrees are what arises when one wants to discuss "treelike" objects without reference to a metric or topology. In this paper, the pretrees are the "primitive" objects and metrics/topologies are additional structures on the pretree — think of it the same way a Riemannian metric is a compatible addition to a manifold's smooth structure.

Fix a set T; an interval function on T is a function $[\cdot,\cdot]\colon T \times T \to \mathcal{P}(T)$, where $\mathcal{P}(T)$ is the power set of T, that satisfies the following axioms: for all $p, q, r \in T$,

1. (symmetric) $[p,q] = [q,p]$ contains $\{p,q\}$;
2. (thin) $[p,r] \subset [p,q] \cup [q,r]$; and
3. (linear) if $r \in [p,q]$ and $q \in [p,r]$, then $q = r$.

A pretree is pair $(T, [\cdot,\cdot])$ of a nonempty set T and an interval function $[\cdot,\cdot]$ on T.

The subsets $[p,q] \subset T$ are called *closed intervals* and they should be thought of as the points between p and q (inclusive). We can similarly define *open* (resp. *half-open*) intervals by excluding both (resp. exactly one) of $\{p,q\}$. Generally, "interval" (with no qualifier) refers to any of the three types of intervals we have defined. An interval is degenerate if it is empty or a singleton. We usually omit the interval function and denote a pretree by T. Note that the real line $\mathbb{R}$ is a pretree.

Any subset $S \subset T$ of a pretree inherits an interval function: $[u,v]_S := [u,v] \cap S$ for all $u, v \in S$. A subset $S \subset T$ is convex if $[p,q] \subset C$ for all $p, q \in C$; or equivalently, $[\cdot,\cdot]_S$ is the restriction of $[\cdot,\cdot]$ to $S \times S \subset T \times T$. A system of pretrees is a set system $\mathcal{T} = \bigsqcup_{j \in J} T_j$ and a disjoint union of interval functions on T_j; we call these systems "pretrees" for short.

Let $(T, [\cdot,\cdot])$ and $(T', [\cdot,\cdot]')$ be pretrees. A pretree-isomorphism $f\colon (T, [\cdot,\cdot]) \to (T', [\cdot,\cdot]')$ is a bijection $f\colon T \to T'$ satisfying $f([p,q]) = [f(p), f(q)]'$ for all $p, q \in T$. Similarly, a pretree-automorphism of $(T, [\cdot,\cdot])$ is a pretree-isomorphism $g\colon (T, [\cdot,\cdot]) \to (T, [\cdot,\cdot])$. A pretree is real if its closed intervals are pretree-isomorphic to closed intervals of $\mathbb{R}$. An arc of a real pretree T is a nonempty union of an ascending chain of nondegenerate intervals. A real pretree is degenerate if it is a singleton; and a system of real pretrees is degenerate if *all* components are degenerate. By definition, the real line $\mathbb{R}$ is a real pretree. Being real is a property of a pretree, not an added structure like a metric!

Fix a real pretree T; a convex pseudometric on T is a function $d\colon T \times T \to \mathbb{R}_{\geq 0}$ satisfying the following axioms: for all $p, q, r \in T$,

1. (symmetric) $d(p,q) = d(q,p)$;
2. (convex) $d(p,r) = d(p,q) + d(q,r)$ if $q \in [p,r]$; and
3. (continuous) $d(p,q) = 2\,d(p,q')$ for some $q' \in [p,q]$.

For any given convex pseudometric d on T, the preimage $d^{-1}(0) \subset T \times T$ is an equivalence relation on the real pretree T and each equivalence class is convex. A convex metric on T is a convex pseudometric whose equivalence relation $d^{-1}(0)$ is the equality relation on T. A (metric) tree (or $\mathbb{R}$-tree) is a real pretree with a convex metric; a forest is a system of trees. For example, the real line $\mathbb{R}$ is a tree with the *standard metric* $d_{\mathrm{std}}(p,q) := |p-q|$. Generally, if d is a convex pseudometric on a real pretree T, then the quotient metric space is a tree (T_d, d) also known as the associated tree.

A λ-homothety of trees $h\colon (T,d) \to (Y,\delta)$ is a pretree-isomorphism $\iota\colon T \to Y$ that uniformly scales the metric d by λ:

$$\delta(h(p),h(q)) = \lambda\, d(p,q) \text{ for all } (p,q) \in \operatorname{dom}(d) = T \times T; \text{ equivalently, } h^*\delta = \lambda d,$$

where $h^*\delta$ is the *pullback of δ via h*. A homothety is a λ-homothety for some $\lambda > 0$; it is expanding (resp. an isometry) if $\lambda > 1$ (resp. $\lambda = 1$). An isometry $\iota\colon (T,d) \to (T,d)$ is elliptic if it fixes a point of T; otherwise, it is loxodromic and acts by a nontrivial translation on its *axis*, the unique ι-invariant arc of (T,d) isometric to $(\mathbb{R}, d_{\mathrm{std}})$; the translation distance $\|\iota\|_d \in \mathbb{R}_{\geq 0}$ is 0 if ι is elliptic and equal to the displacement of points in ι's axis if ι is loxodromic. These definitions extend componentwise to forests.

Let d_1 be a nonconstant convex pseudometric on $\mathcal{T}$ and $d_{i+1}\colon d_i^{-1}(0) \to \mathbb{R}_{\geq 0}$ a (system of) nonconstant convex pseudometric(s) for $1 \leq i < n$. The sequence $(d_i)_{i=1}^n$ will be known as an n-level hierarchy of convex pseudometrics on real pretrees $\mathcal{T}$. We usually say just "hierarchies" for short. A hierarchy $(d_i)_{i=1}^n$ has full support if d_n is a convex metric. A pseudoforest is a pair $(\mathcal{T}, (d_i)_{i=1}^n)$ of real pretrees and a hierarchy with full support; a pseudotree is a pseudoforest with one component. A $(\lambda_i)_{i=1}^n$-homothety of n-level pseudoforests $h\colon (\mathcal{T}, (d_i)_{i=1}^n) \to (\mathcal{Y}, (\delta_i)_{i=1}^n)$ is a system of pretree-isomorphisms $h\colon \mathcal{T} \to \mathcal{Y}$ that scales each pseudometric d_i by λ_i:

$$\delta_i(h(p),h(q)) = \lambda_i\, d_i(p,q) \text{ for all } i \geq 1 \text{ and } (p,q) \in \operatorname{dom}(d_i);$$

a homothety is a $(\lambda_i)_{i=1}^n$-homothety for some $\lambda_i > 0$; it is expanding (resp. isometry) if each $\lambda_i > 1$ (resp. each $\lambda_i = 1$). As with trees, an isometry of a pseudotree is either elliptic (fixes a point) or loxodromic (translates a "pseudoaxes"). Hierarchies and pseudoforests are the fundamental (perhaps novel) tool in this paper. They are first used in Chapter III.

I.3 Simplicial actions and train tracks

For a pretree T, a direction at $p \in T$ is a maximal subset $D_p \subset T$ not separated by p, i.e. $p \notin [q,r]$ for all $q, r \in D_p$. A branch point is a point with at least three directions. An endpoint is a point with at most one direction. A *simple pretree* is a pretree whose closed intervals are finite subsets. A pretree T is simplicial if it is real, its subset V of branch points and endpoints is a simple pretree, and no convex proper subset contains V; a vertex is a point in V. An (open) edge in a simplicial pretree T is a maximal convex subset $e \subset T$ that contains no vertex. By construction, edges are open intervals; the corresponding closed intervals in T are called closed edges.

Remark. Being simplicial is a property of a pretree, not an added structure! Besides that, our definition of a simplicial pretree is more general (with one exception) than the standard definition of a *simplicial tree* and has the advantage that it is independent of any choice of metric/topology. See [Mut21b, Interlude] for a discussion on this distinction. The one exception: the real line $\mathbb{R}$ is not simplicial!

An F*-pretree* is a pretree with an F-action by pretree-automorphisms. An F*-pseudotree* (resp. F*-tree*) is a pair of a real F-pretree and an F*-invariant* hierarchy with full support (resp. F*-invariant* convex metric); equivalently, an F-pseudotree (resp. F-tree) is a pseudotree (resp. tree) with an isometric F-action. An F-pseudotree or F-tree is minimal if the underlying F-pretree has no proper nonempty F-invariant convex subset; in this case, the underly F-pretree has no endpoints. We mostly consider minimal F-pseudotrees or F-trees with trivial arc (pointwise) stabilizers.

Suppose an F-pseudotree $(T, (d_i)_{i=1}^n)$ has trivial arc stabilizers. For any nontrivial subgroup $G \leq F$, the characteristic convex subset (of T) for G is the unique minimal nonempty G-invariant convex subset $T(G) \subset T$. In an F-tree (T, d) with trivial arc stabilizers, the restriction of d to $T(G)$ is a G-invariant convex metric, still denoted d; the minimal G-tree $(T(G), d)$ is the characteristic subtree (of (T, d)) for G.

Remark. We do not really need an isometric action to define characteristic convex subsets and minimality. All we need is the F-action on the real pretree T to be *rigid/non-nesting*: no closed interval is sent properly into itself by the F-action [Mut21b, Section II.2]. While rigid actions are central to [Mut21b], they are superseded by isometric actions in this paper.

An F-pretree T is simplicial if T is simplicial and admits an F-invariant convex metric d; equivalently, a simplicial F-pretree is a simplicial pretree with a rigid F-action. Any simplicial F-pretree has an open cone (over a finite dimensional open simplex) worth of F-invariant convex metrics (up to an equivariant isometry *isotopic* to the identity map). The definitions given so far extend componentwise to systems.

Let $\mathcal{T}$ and $\mathcal{T}'$ be simplicial pretrees and $f\colon \mathcal{T} \to \mathcal{T}'$ a *tight cellular map*, i.e. a function that maps vertices to vertices and the restriction to any closed edge is a *pretree-embedding*, i.e. a pretree-isomorphism onto its image. For any choice of convex metrics d, d' on $\mathcal{T}, \mathcal{T}'$ respectively, there is a unique map $(\mathcal{T}, d) \to (\mathcal{T}', d')$ that is *linear* on edges and *isotopic* to f; whenever a choice of convex metric is made, we implicitly replace f with this map.

Let $\mathcal{T}$ be a free splitting of $\mathcal{F}$, i.e. minimal simplicial $\mathcal{F}$-pretrees with trivial edge stabilizers, and suppose $\psi\colon \mathcal{F} \to \mathcal{F}$ is a free group automorphism. The ψ*-twisted* free splitting $\mathcal{T}\psi$ is the same real pretrees $\mathcal{T}$ but the original simplicial $\mathcal{F}$-action is precomposed with ψ. A (relative) topological representative for ϕ is a ψ*-equivariant* tight cellular map $f\colon \mathcal{T} \to \mathcal{T}$ on a nondegenerate free splitting $\mathcal{T}$ of $\mathcal{F}$: ψ-equivariance means $f(x \cdot p) = \psi(x) \cdot f(p)$ for all $x \in \mathcal{F}$ and $p \in \mathcal{T}$, or equivalently, $f\colon T \to T\phi$ is equivariant. Given a topological representative $f\colon \mathcal{T} \to \mathcal{T}$ for ψ, we let $[f]$ denote the induced map on the quotient $\mathcal{F}\backslash\mathcal{T}$; we say $[f]$ is a *topological representative for the outer class* $[\psi]$. A (relative) train track for ψ is a topological representative $\tau\colon \mathcal{T} \to \mathcal{T}$ for ψ on whose iterates τ^m $(m \geq 1)$ are topological representatives for ϕ^m; or equivalently, whose iterates τ^m restrict to pretree-embeddings on closed edges.

For any free splitting $\mathcal{T}$ of $\mathcal{F}$, Bass-Serre theory gives a uniform bound on the number of $\mathcal{F}$-orbits of edges (linear in $\operatorname{rank}(\mathcal{F})$) and relates the vertices with nontrivial stabilizers in a (componentwise) connected fundamental domain to a (possibly empty) *free factor system* $\mathcal{F}[\mathcal{T}]$ of $\mathcal{F}$ — take this as the working definition of "free factor systems". The theory also gives a uniform bound on the *complexity* (e.g. ranks) of free factor systems. A free factor system $\mathcal{F}[\mathcal{T}]$ *proper* if $\mathcal{F}[\mathcal{T}] \neq \mathcal{F}$; equivalently, $\mathcal{F}[\mathcal{T}]$ is proper if and only if $\mathcal{T}$ is not degenerate. Any proper free factor system of $\mathcal{F}$ has strictly lower complexity than $\mathcal{F}$. The trivial free factor system of $\mathcal{F}$ is the (possibly empty) free factor system consisting of the cyclic $\mathcal{F}$-components; it is always proper since we assume $\mathcal{F}$ has a noncyclic component.

Remark. We will abuse notation and write $\mathcal{F}[\mathcal{T}] = \mathcal{F}[\mathcal{T}']$ for two free splittings $\mathcal{T}, \mathcal{T}'$ of $\mathcal{F}$ when we mean: an element of $\mathcal{F}$ is $\mathcal{T}$-elliptic if and only if it is $\mathcal{T}'$-elliptic.

Fix an automorphism $\psi\colon \mathcal{F} \to \mathcal{F}$ and a topological representative $f\colon \mathcal{T} \to \mathcal{T}$ for ψ. By ψ-equivariance of f, the proper free factor system $\mathcal{F}[\mathcal{T}]$ is $[\psi]$-*invariant* — again, we can take this as the definition of $[\psi]$-invariance for proper free factor systems. Form a nonnegative integer square matrix $A[f]$ whose rows and columns are indexed by the $\mathcal{F}$-orbits of edges in $\mathcal{T}$; and the entry at row-$[e]$ and column-$[e']$ is given by the number of e-translates in the interval $f(e')$, where e, e' are edges in T. The topological representative f is irreducible if the matrix $A[f]$ is *irreducible*; or equivalently, if, for any pair of edges e, e' in $\mathcal{T}$, a translate of e is contained $f^m(e')$ for some $m = m(e, e') \geq 1$. It is a foundational theorem of Bestvina–Handel that automorphisms have irreducible train tracks.

Theorem I.1 (cf. [BH92, Theorem 1.7])**.** *Let $\psi\colon \mathcal{F} \to \mathcal{F}$ be an automorphism of a free group system and $\mathcal{Z}$ a $[\psi]$-invariant proper free factor system of $\mathcal{F}$. Then there is an irreducible train track $\tau\colon \mathcal{T} \to \mathcal{T}$ for ψ, where the components of $\mathcal{Z}$ are $\mathcal{T}$-elliptic.*

The proof outline of [Mut21b, Theorem I.1] explains how to deduce the theorem as currently stated from the cited theorem.

Suppose $\psi\colon \mathcal{F} \to \mathcal{F}$ is an automorphism with an irreducible topological representative $f\colon \mathcal{T} \to \mathcal{T}$. Perron–Frobenius theory implies the matrix $A[f]$ has a unique real eigenvalue $\lambda = \lambda[f] \geq 1$ with a unique positive left eigenvector $\nu[f]$ whose entries sum to 1. From the eigenvector $\nu[f]$, we get an $\mathcal{F}$-invariant convex metric d_f on $\mathcal{T}$ (unique up to an equivariant isometry isotopic to the identity map). The restriction of $f\colon (\mathcal{T}, d_f) \to (\mathcal{T}, d_f)$ to any edge is a λ-*homothetic embedding*; the metric d_f is the eigenmetric (on $\mathcal{T}$) for $[f]$. If $\lambda = 1$, then f is a ψ-equivariant *simplicial automorphism* of $\mathcal{T}$.

I.4 Growth types and limit trees

Since the introduction of train tracks, it has been standard to construct *limit forests* by iterating an expanding irreducible train track (Section II.1). Unfortunately, such a construction is not canonical as it can depend on the initial train track. The main idea of the paper: patch together a "descending" sequence of limit trees to get a limit pseudoforest and inductively "normalize" its hierarchy into a canonical limit pseudoforest. We immediately have an obstacle to this suggested strategy: Bestvina–Handel's theorem does not guarantee the existence of an expanding irreducible train track. We first generalize the construction of limit trees to deal with this obstacle (Section II.4).

Fix a free group system $\mathcal{G}$ of finite type (unlike $\mathcal{F}$, all components $\mathcal{G}$ can be cyclic), an automorphism $\psi\colon \mathcal{G} \to \mathcal{G}$, and a metric free splitting $(\mathcal{T}, d)$ whose free factor system $\mathcal{F}[\mathcal{T}]$ is $[\psi]$-invariant. An element $x \in \mathcal{G}$ $[\psi]$-grows exponentially rel. d with rate $\lambda_x > 1$ if it is $\mathcal{T}$-loxodromic and the sequence $\left(m^{-1} \log \|\psi^m(x)\|_d\right)_{m\geq 0}$ converges to $\log \lambda_x$. If an element x $[\psi]$-grows exponentially rel. d with rate λ_x, then it $[\psi]$-grows exponentially rel. d' with rate λ_x for any metric free splitting $(\mathcal{T}', d')$ of $\mathcal{G}$ with $\mathcal{F}[\mathcal{T}'] = \mathcal{F}[\mathcal{T}]$; set $\mathcal{Z} \coloneqq \mathcal{F}[\mathcal{T}]$ and say the element $[\phi]$-grows exponentially rel. $\mathcal{Z}$. An element $x \in \mathcal{G}$ $[\psi]$-grows polynomially rel. $\mathcal{Z}$ with degree $< n$ if the sequence $(m^{-n}\|\psi^m(x)\|_d)_{n\geq 0}$ converges to 0. By [Mut21b, Proposition III.1], any element of $\mathcal{G}$ $[\psi]$-grows either exponentially or polynomially rel. $\mathcal{Z}$. The *growth type* of a conjugacy class $[x]$ is well-defined and preserved when passing to invariant subgroup systems of finite type.

The automorphism ψ is exponentially growing rel. $\mathcal{Z}$ if some element $[\psi]$-grows exponentially rel. $\mathcal{Z}$; otherwise, ψ is polynomially growing rel. $\mathcal{Z}$. The growth type of an outer class $[\psi]$ is also well-defined. The "rel. $\mathcal{Z}$" in our terminology may be omitted when $\mathcal{Z}$ is the trivial free factor system of $\mathcal{G}$. The next proposition deals with the first obstacle:

Proposition I.2 (cf. [Mut21b, Proposition III.2]). *Let $\psi\colon \mathcal{F} \to \mathcal{F}$ be an automorphism of a free group system. Then there is a:*

1. *a minimal $\mathcal{F}$-forest $(\mathcal{Y}, \delta)$ with trivial arc stabilizers; and*
2. *a unique ψ-equivariant expanding homothety $h\colon (\mathcal{Y}, \delta) \to (\mathcal{Y}, \delta)$.*

The underlying pretrees $\mathcal{Y}$ are degenerate if and only if ψ is polynomially growing.

If $[\psi]$ is polynomially growing, then the *limit forest for* $[\psi]$ is the degenerate $\mathcal{F}$-forest. Otherwise, $[\psi]$ is exponentially growing and the constructed nondegenerate $\mathcal{F}$-forest $(\mathcal{Y}, \delta)$ is the *limit forest for* $[\psi]$ *rel.* $\mathcal{Z}$, for some $[\psi]$-invariant proper free factor system $\mathcal{Z}$ (see Sections II.1 and II.4). Unfortunately, these limit forests still depend on the choice of $\mathcal{Z}$. Our goal is to give a universal contruction.

Given the central tool (hierarchies) and objective (universal limit trees), we outline again how these two fit together. Gaboriau–Levitt's index theory [GL95] gives a uniform bound on the complexity of the "point stabilizers system" $\mathcal{G}[\mathcal{Y}]$ for a minimal $\mathcal{F}$-forest $(\mathcal{Y}, \delta)$ with trivial arc stabilizers — this is a partial generalization of Bass–Serre theory. When $\mathcal{Y}$ is not degenerate, the subgroup system $\mathcal{G}[\mathcal{Y}]$ has strictly lower *complexity* than $\mathcal{F}$. This allows us to induct on complexity (see Chapter III).

Suppose the automorphism $\psi\colon \mathcal{F} \to \mathcal{F}$ has a nondegenerate limit forest $(\mathcal{Y}_1, \delta_1)$ with nontrivial point stabilizers; the system of stabilizer $\mathcal{G} \coloneqq \mathcal{G}[\mathcal{Y}]$ has strictly smaller complexity than $\mathcal{F}$. By ψ-equivariance of $h_1\colon \mathcal{Y}_1 \to \mathcal{Y}_1$, the $\mathcal{F}$-orbits of points with nontrivial stabilizers are permuted by $[h_1]$, the system $\mathcal{G}$ is $[\psi]$-invariant, and the "restriction" of $[\psi]$ to $\mathcal{G}$ determines a unique outer automorphism $[\varphi]$ of $\mathcal{G}$.

Suppose $\varphi\colon \mathcal{G} \to \mathcal{G}$ has a nondegenerate limit forest $(\mathcal{Y}_2, \delta_2)$. Using the blow-up construction from [Mut21b], we equivariantly blow-up $\mathcal{Y}_1$ with respect to $h_i\colon \mathcal{Y}_i \to \mathcal{Y}_i$ $(i = 1, 2)$ to get real pretrees $\mathcal{T}$ with a minimal rigid $\mathcal{F}$-action and a ψ-equivariant "$\mathcal{F}$-expanding" pretree-isomorphism $f\colon T \to T$ induced by h_1 and h_2. In fact, the blow-up construction implies the $\mathcal{F}$-pretrees $\mathcal{T}$ inherit an $\mathcal{F}$-invariant 2-level hierarchy (δ_1, δ_2) with full support and f is an expanding homothety with respect to this hierarchy. So we have a *limit pseudoforest* $(\mathcal{T}, (\delta_1, \delta_2))$ for $[\psi]$ (see Section III.1). Under what conditions can we construct an $\mathcal{F}$-invariant convex metric on $\mathcal{T}$ from (δ_1, δ_2)? The heart of the paper is the following observation: the two limit forests $(\mathcal{Y}_i, \delta_i)$ are paired with *attracting laminations* $\mathcal{L}_i^+[\psi]$ partially ordered by inclusion; an $\mathcal{F}$-invariant convex metric on $\mathcal{T}$ can be naturally constructed from (δ_1, δ_2) if $\mathcal{L}_2^+[\psi]$ is not contained in $\mathcal{L}_1^+[\psi]$ (see Section III.4)!

I.5 Bounded cancellation and laminations

Let $(\mathcal{T}, d)$ be metric free splitting of $\mathcal{F}$, $(\mathcal{Y}, \delta)$ a minimal $\mathcal{F}$-forest with trivial arc stabilizers, and denote the interval functions by $[\cdot,\cdot]_T$ and $[\cdot,\cdot]_Y$ respectively. A map $f\colon (\mathcal{T}, d) \to (\mathcal{Y}, \delta)$ is piecewise-linear (PL) if the restriction to any closed edge is a linear map; an equivariant PL-map exists if and only if $\mathcal{T}$-elliptic elements in $\mathcal{F}$ are $\mathcal{Y}$-elliptic. Equivariant PL-maps are surjective and Lipschitz since the isometric $\mathcal{F}$-action on $(\mathcal{Y}, \delta)$ is minimal and there are only finitely many $\mathcal{F}$-orbits of edges in $\mathcal{T}$; 1-Lipschitz maps are also known as metric maps. A PL-map is simplicial if its target is a metric free splitting. Generally, if $\mathcal{T}, \mathcal{Y}$ are free splittings of $\mathcal{F}$, then an equivariant function $f\colon \mathcal{T} \to \mathcal{Y}$ is a (simplicial) PL-map if its restrictions to any closed edge is isotopic to a linear map with respect to some/any $\mathcal{F}$-invariant convex metrics d, δ on $\mathcal{T}, \mathcal{Y}$ respectively.

Lemma I.3 (bounded cancellation)**.** *Let $f\colon (\mathcal{T}, d) \to (\mathcal{Y}, \delta)$ be an equivariant PL-map. For some constant $C[f] \geq 0$ and all points $p, q \in \mathcal{T}$, the image $f([p,q]_T)$ is in the $C[f]$-neighbourhood of the interval $[f(p), f(q)]_Y$.*

This lemma and proof is due to Bestvina–Feighn–Handel [BFH97, Lemma 3.1]; the proof works verbatim if the assumption on $(\mathcal{Y}, \delta)$ is relaxed to a minimal "very small" $\mathcal{F}$-forest. We call any such $C[f]$ a cancellation constant for f.

Sketch of proof. Let $\mathrm{Lip}(f)$ be the Lipschitz constant and $\mathrm{vol}(\mathcal{T}, d)$ the *volume* (mod $\mathcal{F}$). Then $f = g \circ h$ for some equivariant $\mathrm{Lip}(f)$-homothety h and metric PL-map g. Suppose f is simplicial. Then g factors as finitely many equivariant edge collapses and Stallings folds followed by an equivariant metric homeomorphism. The homeomorphism and each edge collapse have cancellation constants 0. A fold has a cancellation constant given by the length of folded segment. Finally, the metric PL-map g has a cancellation constant since cancellation constants are (sub)additive over compositions of metric maps. As cancellation constants are preserved by precomposition with homeomorphisms, the PL-map $f = g \circ h$ has a cancellation constant $C[f] < \mathrm{Lip}(f)\,\mathrm{vol}(\mathcal{T}, d)$.

Otherwise, the PL-map f is not simplicial. For a contradiction, suppose the image $f([p,q]_T)$ is not in the $\mathrm{Lip}(f)\,\mathrm{vol}(\mathcal{T}, d)$-neighbourhood of $[f(p), f(q)]_Y$ for some $p, q \in \mathcal{T}$. Let $\delta(f(r_0), [f(p), f(q)]_Y) > \mathrm{Lip}(f)\,\mathrm{vol}(\mathcal{T}, d) + \epsilon_0$ for some $\epsilon_0 > 0$ and point $r_0 \in [p,q]_T$. For any $\epsilon > 0$, the PL-map f can be *approximated* by an equivariant simplicial PL-map f_ϵ with $\mathrm{Lip}(f_\epsilon) < \mathrm{Lip}(f) + \epsilon$ and $C[f_\epsilon] \geq \mathrm{Lip}(f)\,\mathrm{vol}(\mathcal{T}, d) + \epsilon_0$ (see [BF94, Theorem 2.2] or [Hor17, Theorem 6.1]). By the previous paragraph, $C[f_\epsilon] < \mathrm{Lip}(f_\epsilon)\,\mathrm{vol}(\mathcal{T}, d)$ for $\epsilon > 0$. So $C[f_\epsilon] < \mathrm{Lip}(f)\,\mathrm{vol}(\mathcal{T}, d) + \epsilon_0$ for small enough $\epsilon > 0$ — a contradiction. □

Remark. The results in this section apply to ψ-equivariant PL-maps $g\colon (\mathcal{T}, d) \to (\mathcal{T}, d)$ for any automorphism $\psi\colon \mathcal{F} \to \mathcal{F}$: view g as an equivariant PL-map $(\mathcal{T}, d) \to (\mathcal{T}\psi, d)$ instead.

A line in a forest is an arc that is isometric to $(\mathbb{R}, d_{\mathrm{std}})$; a ray in a forest is an arc that is isometric to $(\mathbb{R}_{\geq 0}, d_{\mathrm{std}})$ and its origin is the point corresponding to 0 under the isometry. Two rays are *end-equivalent* if their intersection is a ray; an end of a forest is an end-equivalence class of rays in the forest. Note that there is a natural bijection between the set of lines in a forest and set of unordered pairs of distinct ends in the forest. For simplicial $\mathcal{F}$-pretrees $\mathcal{T}$, the notions of line/ray/end are well-defined for the cone of $\mathcal{F}$-invariant convex metrics on $\mathcal{T}$.

Corollary I.4 (cf. [GJLL98, Lemma 3.4])**.**
Let $f\colon (\mathcal{T}, d) \to (\mathcal{Y}, \delta)$ be an equivariant PL-map.

1. *For any ray ρ in $(\mathcal{T}, d)$ with origin p_0, the image $f(\rho)$ is either bounded or in the $C[f]$-neighbourhood of a unique ray $f_*(\rho) \subset f(\rho)$ with origin $f(p_0)$.*
2. *For any line l in $(\mathcal{T}, d)$, if both ends of l have unbounded f-images, then $f(l)$ is in a $C[f]$-neighbourhood of a unique line $f_*(l) \subset f(l)$.*
3. *For any end ϵ of $(\mathcal{Y}, \delta)$, there is a unique end $f^*(\epsilon)$ of $(\mathcal{T}, d)$ with $\epsilon = f_*(f^*(\epsilon))$.*

Sketch of proof.
(1): Let ρ be a ray in $(\mathcal{T}, d)$, $p_0 \in \rho$ its origin, $s_0 = f(p_0)$, and $f(\rho)$ unbounded. Use Figure 1 for reference. By bounded cancellation and the Lipschitz property, $f(\rho)$ has at most one end of $(\mathcal{Y}, \delta)$. For some $n \geq 0$, assume $s_n \in [s_0, f(p)]_Y$ for all $p \in \rho \setminus [p_0, p_n]_T$. Since $f(\rho)$ is unbounded,

$$\delta(s_0, f(p_{n+1})) > 2\,\delta(s_0, s_n) + C \quad (\text{set } C \coloneqq C[f])$$

for some $p_{n+1} \in \rho \setminus [p_0, p_n]_T$. Pick $s_{n+1} \in [s_0, f(p_{n+1})]_Y$ satisfying $\delta(s_0, s_{n+1}) > 2\,\delta(s_0, s_n)$ and $\delta(s_{n+1}, f(p_{n+1})) > C$; so $s_n \in [s_0, s_{n+1}]_Y$. By bounded cancellation, the interval $[s_0, s_{n+1}]_Y \subset f([p_0, p_{n+1}]_T)$ is disjoint from $f(\rho \setminus [p_0, p_{n+1}]_T)$. So the union $\bigcup_{n\geq 0}[s_0, s_n]_Y$ is a ray $f_*(\rho)$ in $f(\rho)$ with origin s_0. By construction, $f(\rho)$ is in the C-neighbourhood of $f_*(\rho)$. Any bounded neighbourhood of a ray contains a unique ray, up to end-equivalence.

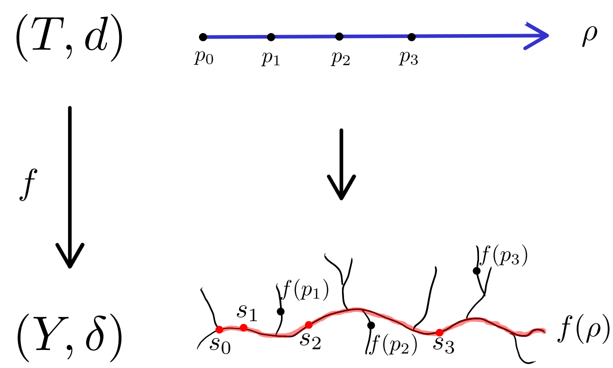

Figure 1: The ray $f_*(\rho)$ with origin $s_0 = f(p_0)$ is built inductively.

(2): Represent both ends of l with rays $\rho^{\pm} \subset l$ with the same origin. By Part (1) and bounded cancellation, we have rays $f_*(\rho^{\pm})$ representing unique distinct ends $\epsilon^{\pm}$ of $(\mathcal{Y}, \delta)$; moreover, $f(l) = f(\rho^- \cup \rho^+)$ is in the $C[f]$-neighbourhood of $f_*(\rho^-) \cup f_*(\rho^+) \subset f(l)$. Let $f_*(l) \subset f_*(\rho^-) \cup f_*(\rho^+)$ be the line determined by the ends $\epsilon^{\pm}$. Then $f(l)$ is in the $C[f]$-neighbourhood of $f_*(l)$. Any bounded neighbourhood of a line contains a unique line.

(3): The argument is almost the same. Let ρ' be a ray representing ϵ and $s_0 = q_0$ its origin. Pick points $q_{n+1}, s_{n+1} \in \rho'$ with $\delta(s_0, s_{n+1}) > 2\,\delta(s_0, s_n)$, $\delta(s_0, q_{n+1}) > 2\,\delta(s_0, s_n) + C$, and $\delta(s_{n+1}, q_{n+1}) > C$. Since $f\colon \mathcal{T} \to \mathcal{Y}$ is surjective, we can lift q_n to $p_n \in T$. By bounded cancellation and K-Lipschitz property, the distance $d(p_0, [p_n, p_{n+m}]_T) > \frac{1}{K}\delta(s_0, s_n) > 0$. Thus $(p_n)_{n \geq 0}$ determines an end e of $(\mathcal{T}, d)$ with unbounded f-image. Let ρ be a ray representing e with origin p_0. As $\rho' \subset f(\rho)$ by construction, we get $f_*(\rho) = \rho'$ by Part (1). By Part (2), the end e is the unique end with $f_*(e) = \epsilon$ and we denote it by $f^*(\epsilon)$. □

The corollary defines equivariant lifting (resp. projecting) function f^* (resp. f_*), where the domain $\mathrm{dom}(f^*)$ of f^* is the set of lines in $(\mathcal{Y}, \delta)$ and the domain $\mathrm{dom}(f_*)$ of f_* is the set of lines in $(\mathcal{T}, d)$ whose ends both have unbounded f-images. Note that the image $\mathrm{im}(f^*)$ is $\mathrm{dom}(f_*)$; moreover, f^* and f_* are inverses of each other. Both lifting and projecting functions are *canonical*: $f^* = g^*$ and $f_* = g_*$ for any equivariant maps $f, g\colon (\mathcal{T}, d) \to (\mathcal{Y}, \delta)$ since f, g will be a bounded δ-distance from an equivariant PL-map; for lack of better notation, we still denote the functions by f^*, f_* despite this independence.

Alternatively, we view f^* and f_* as functions on the sets of $\mathcal{F}$-orbits of lines. We can equip these sets with a natural topology. The set $\mathbb{R}(\mathcal{Y}, \delta)$ of $\mathcal{F}$-orbits of lines in $(\mathcal{Y}, \delta)$ has the following topology: for any $p, q \in \mathcal{Y}$, let $U[p, q]$ be the $\mathcal{F}$-orbit of lines that contain a translate of $[p, q]$; the collection $\{U[p, q] : p, q \in \mathcal{Y}\}$ is a basis for the space of ($\mathcal{F}$-orbits of) lines. This space is well-defined for the equivariant homothetic class of $(\mathcal{Y}, \delta)$. The space of lines is also well-defined for the free splitting $\mathcal{T}$ and denoted $\mathbb{R}(\mathcal{T})$.

Claim I.5. *The canonical lifting function $f^*\colon \mathbb{R}(\mathcal{Y}, \delta) \to \mathbb{R}(\mathcal{T})$ is a topological embedding.*

Henceforth, we identify $\mathbb{R}(\mathcal{Y},\delta)$ with a subspace of $\mathbb{R}(\mathcal{T})$ using the canonical embedding f^*.

Sketch of proof. We first prove the injection f^* is continuous. Let $\Lambda \subset \mathbb{R}(\mathcal{T})$ be a closed subset and $\Lambda_f \coloneqq \Lambda \cap \mathrm{im}(f^*)$. Suppose $[\gamma]$ is in the closure of $f_*(\Lambda_f)$ in $\mathbb{R}(\mathcal{Y},\delta)$. For continuity, it is enough to show $f^*[\gamma] \in \Lambda$. Fix a long interval $I_\gamma \subset \gamma$; then $I_\gamma \subset [f(p), f(q)]_Y$ for some $p, q \in f^*(\gamma)$. As $[\gamma]$ is in the closure of $f_*(\Lambda_f)$, the interval $I_\gamma \subset \gamma$ is in the line $f_*(l)$ for some $[l] \in \Lambda_f$. By bounded cancellation, the f-image of the intersection $I_l \coloneqq f^*(\gamma) \cap l$ contains a long interval in I_γ. As the interval I_γ exhausts γ, the interval I_l exhausts $f^*(\gamma)$; in particular, any interval in $f^*(\gamma)$ is contained in l for some $[l] \in \Lambda$. So $f^*[\gamma]$ is in the closed subset Λ.

We finally prove $f^*\colon \mathbb{R}(\mathcal{Y},\delta) \to \mathrm{im}(f^*)$ is an open map — the image $\mathrm{im}(f^*) \subset \mathbb{R}(\mathcal{T})$ has the subspace topology. Suppose $p, q \in \mathcal{Y}$ and $[\gamma] \in U[p,q]$, i.e. a line γ in $(\mathcal{Y},\delta)$ contains an interval $[p,q]_Y$. There is an interval $[u,v]_T \subset f^*(\gamma)$ whose f-image covers a long neighbourhood of $[p,q]_Y$. By bounded cancellation, any line $f^*(\gamma')$ containing $[u,v]_T$ will have an f_*-image γ' containing $[p,q]_Y$. So $f^*[\gamma] \in U[u,v] \cap \mathrm{im}(f^*) \subset f^*(U[p,q])$. As $[\gamma] \in U[p,q]$ was arbitrary, the image $f^*(U[p,q])$ is open in $\mathrm{im}(f^*)$. □

Now assume $\mathcal{T}'$ is a free splitting of $\mathcal{F}$ with $\mathcal{F}[\mathcal{T}] = \mathcal{F}[\mathcal{T}']$ and let $f\colon \mathcal{T} \to \mathcal{T}'$ be an equivariant PL-map. The folds in the factorization of f never identify points in the same $\mathcal{F}$-orbit. For $[l] \in \mathbb{R}(\mathcal{T})$, each end of l has unbounded f-image, i.e. $\mathrm{dom}(f_*) = \mathbb{R}(\mathcal{T})$; so $f_*\colon \mathbb{R}(\mathcal{T}) \to \mathbb{R}(\mathcal{T}')$ is a canonical homeomorphism (with inverse f^*). Similarly, if $g\colon \mathcal{T} \to \mathcal{T}$ is a ψ-equivariant PL-map for some automorphism $\psi\colon \mathcal{F} \to \mathcal{F}$, then $g_*\colon \mathbb{R}(\mathcal{T}) \to \mathbb{R}(\mathcal{T})$ are canonical homeomorphisms for $[\psi]$.

Remark. We use ambiguous terminology and say "line" when we mean a line or an $\mathcal{F}$-orbit of a line; our notation remains distinct: "l" is always a line, while "$[l]$" is its $\mathcal{F}$-orbit.

A lamination in $(\mathcal{Y},\delta)$ (resp. $\mathcal{T}$) is a nonempty closed subset of $\mathbb{R}(\mathcal{Y},\delta)$ (resp. $\mathbb{R}(\mathcal{T})$); when the $\mathcal{F}$-forest in question is clear, we say "lamination" with no qualifier. An element of a lamination is called a leaf; a leaf segment of a lamination Λ is a nondegenerate closed interval in a line representing a leaf of Λ. A lamination is minimal if each leaf is dense in the lamination; a lamination is perfect if it has no isolated leaves.

Let $[l]$ be a line and Λ a lamination in $\mathbb{R}(\mathcal{Y},\delta)$ (or $\mathbb{R}(\mathcal{T})$). A sequence $[l_m]_{m\geq 0}$ in the space of lines weakly limits to $[l]$ if some subsequence converges to $[l]$; we say $[l]$ is a *weak limit* of the sequence. The sequence $[l_m]_{m\geq 0}$ weakly limits to Λ if it weakly limits to every leaf of Λ. The "weak" terminology is used to highlight that the space of lines is not Hausdorff — a convergent sequence may have multiple limits!

More generally, a sequence $[p_m, q_m]_{m\geq 0}$ of intervals *converges* to $[l]$ if, for any closed interval $[a,b] \subset l$, $[p_m, q_m]$ contains a translate of $[a,b]$ for $n \gg 1$ (i.e. for large enough n) — precisely, there is an $M[a,b] \geq 1$ such that $U[a,b]$ contains $U[p_m, q_m]$ for $m \geq M[a,b]$. Again, a sequence of intervals weakly limits to $[l]$ if some subsequence converges to $[l]$ and it weakly limits to Λ if it weakly limits to every leaf of Λ.

II Limit forests

In this chapter, we sketch the proof of Proposition I.2 (existence of limit forests) and, in the process, introduce stable laminations. The first half deals with limit forests for expanding irreducible train tracks; then, in the second half, we extend the results to all limit forests.

II.1 Constructing limit forests (1)

This is a summary of [GLL98, Appendix]; the reader is invited to read that appendix.

Fix an automorphism $\psi\colon \mathcal{F} \to \mathcal{F}$ with an expanding irreducible train track $\tau\colon \mathcal{T} \to \mathcal{T}$. Set $\lambda := \lambda[\tau] > 1$ and let d_τ be the eigenmetric on $\mathcal{T}$ for $[\tau]$. For $m \geq 0$, let d_m^* be the pullback of $\lambda^{-m} d_\tau$ via τ^m:

$$d_m^*(p,q) := \lambda^{-m} d_\tau(\tau^m(p), \tau^m(q)) \leq d_\tau(p,q) \quad \text{for } p, q \text{ in a component of } T.$$

By definition, the pullback d_m^* is an $\mathcal{F}$-invariant (not convex) pseudometric on $\mathcal{T}$ whose quotient metric space is equivariantly isometric to $(\mathcal{T}\psi^m, \lambda^{-m} d_\tau)$. The λ-Lipschitz property for τ with respect to d_τ implies the sequence of pseudometrics d_m^* is (pointwise) monotone decreasing. The limit d_∞ is an $\mathcal{F}$-invariant pseudometric on $\mathcal{T}$, the quotient metric space $(\mathcal{T}_\infty, d_\infty)$ is an $\mathcal{F}$-forest, and the ψ-equivariant λ-Lipschitz train track τ induces a ψ-equivariant λ-homothety $h\colon (\mathcal{T}_\infty, d_\infty) \to (\mathcal{T}_\infty, d_\infty)$; in particular, the equivariant metric surjection $\pi\colon (\mathcal{T}, d_\tau) \to (\mathcal{T}_\infty, d_\infty)$ *semiconjugates* τ to h: $\pi \circ \tau = h \circ \pi$.

As τ is a train track, the restriction of π to any edge of $\mathcal{T}$ is an isometric embedding. So $\mathcal{T}_\infty$ is not degenerate (i.e. not a singleton). In fact, the π-image of any edge of $\mathcal{T}$ can be extended to an axis for a $\mathcal{T}_\infty$-loxodromic elements in $\mathcal{F}$. Thus the $\mathcal{F}$-forest $(\mathcal{T}_\infty, d_\infty)$ is minimal and the uniqueness of h follows from [CM87, Theorem 3.7]. Finally, the minimal $\mathcal{F}$-forest $(\mathcal{T}_\infty, d_\infty)$ has trivial arc stabilizers. This sketches the first case of Proposition I.2. The $\mathcal{F}$-forest $(\mathcal{Y}_\tau, d_\infty) := (\mathcal{T}_\infty, d_\infty)$ is the <u>(forward) limit forest for $[\tau]$</u>.

II.2 Stable laminations (1)

The first part of this section is mostly adapted from Section 1 of [BFH97]. The following definition of "stable" laminations is from [BFH97, Definition 1.3].

Fix an automorphism $\psi\colon \mathcal{F} \to \mathcal{F}$ with an expanding irreducible train track $\tau\colon \mathcal{T} \to \mathcal{T}$. Set $\lambda := \lambda[\tau] > 1$ and pick an edge $e \subset \mathcal{T}$. Expanding irreducibility implies the interval $\tau^k(e)$ contains at least three translates of e for some $k \geq 1$. By the intermediate value theorem, $\tau^k(p) = x \cdot p$ for some $x \in \mathcal{F}$ and $p \in e$. Recall that edges are open intervals; since the restriction of $x^{-1} \cdot \tau^k$ to the edge e is an expanding λ^k-homothetic embedding $e \to \mathcal{T}$ that fixes p and has e in its image, we can extend e to a line $l_p \subset \mathcal{T}$ by iterating $x^{-1} \cdot \tau^k$. By construction, the restriction of $x^{-1} \cdot \tau^k$ to l_p is a λ^k-homothety $l_p \to l_p$ with respect to the eigenmetric d_τ for $[\tau]$; the $\mathcal{F}$-orbit $[l_p]$ is an <u>eigenline</u> of $[\tau^k]$ based at $[p]$ (in $\mathcal{F}\backslash\mathcal{T}$). A <u>stable $\mathcal{T}$-lamination Λ^+ for $[\tau]$</u> is the closure of an eigenline of $[\tau^k]$ for some

$k \geq 1$. By ϕ-equivariance of τ, the restriction of τ to l representing a leaf of a stable lamination Λ^+ is a λ-homothetic embedding. In fact, $[\tau]$ maps eigenlines to eigenlines and the image $\tau_*(\Lambda^+) \coloneqq \{[\tau(l)] : [l] \in \Lambda^+\}$ is also a stable lamination for $[\tau]$.

As the transition matrix $A[\tau]$ is irreducible, it is a block transitive permutation matrix and the "first return" matrix for each block is *primitive*, i.e. has a positive power. There is a bijective correspondence between the stable laminations for $[\tau]$ and the blocks of $A[\tau]$. In particular, there are finitely many stable laminations for $[\tau]$, these laminations are pairwise disjoint, and τ_* transitively permutes them [BFH97, Lemma 1.2]. By finiteness, their union $\mathcal{L}^+[\tau]$ is a lamination and is called the *system of stable laminations* for $[\tau]$.

II.2.1 Quasiperiodic lines

A line $[l]$ in a $\mathcal{F}$-forest is *periodic* if it is the axis for the conjugacy class of some loxodromic element of $\mathcal{F}$. A line $[l]$ is quasiperiodic in an $\mathcal{F}$-forest if any closed interval I in l has an assigned number $L(I) \geq 0$ such that any interval in l of length $L(I)$ contains a translate of I; periodic lines are quasiperiodic. If $[l]$ is a quasiperiodic line, then any leaf of its closure Λ is quasiperiodic and hence dense in Λ (exercise), i.e. Λ is minimal. If $[l]$ is quasiperiodic but not periodic, then no leaf of its closure Λ is isolated (exercise), i.e. Λ is also perfect.

Remark. When the F-action on a free splitting T is free, then our definition of quasiperiodicity is equivalent to [BFH97, Definition 1.7]. However, our definition is weaker when the action is not free. It is the weaker condition that is useful in our setting.

Lemma II.1 (cf. [BFH97, Proposition 1.8])**.** *The eigenlines of $[\tau^k]$ are quasiperiodic but not periodic for $k \geq 1$. Thus the stable laminations for $[\tau]$ are minimal and perfect.*

Proof. There is a length L_0 such that any interval in $\mathcal{T}$ of length L_0 contains an edge. Fix an $\mathcal{F}$-orbit $[I]$ of intervals in an eigenline $[l]$ of $[\tau^k]$ and let $\lambda \coloneqq \lambda[\tau] > 1$. By construction, I is contained in $\tau^{km}(e)$ for some edge e in $\mathcal{T}$ and integer $m \geq 0$. As the blocks in $A[\tau^k]$ are primitive, there is an integer $m' \geq 1$ such that $\tau^{km'}(e')$ contains a translate of e for any edge e' in l. Altogether, any interval in l of length $\lambda^{k(m+m')}L_0$ contains a translate of I. This proves quasiperiodicity.

Now assume, for a contradiction, that the eigenline $[l]$ were periodic, i.e. l is an axis for a $\mathcal{T}$-loxodromic element $x \in \mathcal{F}$. By construction, the $\mathcal{F}$-orbit $[l]$ is τ^k-invariant and hence the cyclic subgroup $\langle x \rangle$ is $[\psi^k]$-invariant. So x is $[\psi]$-periodic as ψ is an automorphism; yet x $[\psi]$-grows exponentially since its axis is an eigenline and τ is expanding. □

Fix an equivariant PL-map $f\colon (\mathcal{T}, d) \to (\mathcal{Y}, \delta)$ and canonically embed $\mathbb{R}(\mathcal{Y}, \delta)$ into $\mathbb{R}(\mathcal{T})$ via f^* (Claim I.5). If a quasiperiodic line $[l] \in \mathbb{R}(\mathcal{T})$ is in the subspace $\mathbb{R}(\mathcal{Y}, \delta) = \mathrm{im}(f^*)$, then so its closure Λ in $\mathbb{R}(\mathcal{T})$ (exercise). Returning to limit forests, the equivariant metric PL-map $\pi\colon (\mathcal{T}, d_\tau) \to (\mathcal{Y}_\tau, d_\infty)$ restricts to an isometric embedding on the leaves of $\mathcal{L}^+[\tau]$; therefore, the stable lamination $\mathcal{L}^+[\tau]$ is in $\mathbb{R}(\mathcal{Y}_\tau, d_\infty) \subset \mathbb{R}(\mathcal{T})$.

II.2.2 Characterizing loxodromics

Let $(\mathcal{Y}_\tau, d_\infty)$ be the limit forest for $[\tau]$, $h : (\mathcal{Y}_\tau, d_\infty) \to (\mathcal{Y}_\tau, d_\infty)$ the unique ψ-equivariant λ-homothety, and $\pi\colon (\mathcal{T}, d_\tau) \to (\mathcal{Y}_\tau, d_\infty)$ the constructed equivariant metric PL-map. By Lemma I.3, the map $\tau\colon \mathcal{T} \to \mathcal{T}$ has a cancellation constant $C \coloneqq C[\tau, d_\tau]$ with respect to d_τ. Set $C' \coloneqq \frac{C}{\lambda - 1}$ and denote the interval functions for $\mathcal{T}$ and $\mathcal{Y}_\tau$ by $[\cdot,\cdot]$ and $[\cdot,\cdot]_\infty$ respectively. The sequence of equivariant metric maps $\tau^m\colon (\mathcal{T}, d_\tau) \to (\mathcal{T}\psi^m, \lambda^{-m} d_\tau)$ have cancellation constants $\sum_{i=1}^m \lambda^{-i} C \leq C'$; so their *limit* π has a cancellation constant $C[\pi] \coloneqq C'$.

Let $P \subset \mathcal{Y}_\tau$ be the $\mathcal{F}$-orbit representatives of points with nontrivial stabilizers. Define the subgroup system $\mathcal{G}[\mathcal{Y}_\tau] \coloneqq \bigsqcup_{p \in P} G_p$, where $G_p \coloneqq \mathrm{Stab}_{\mathcal{F}}(p)$ is the stabilizer in $\mathcal{F}$ of $p \in P$. As the action on $\mathcal{Y}_\tau$ has trivial arc stabilizers, the system $\mathcal{G}[\mathcal{Y}_\tau]$ is malnormal: each component is malnormal (as a subgroup of the appropriate component of $\mathcal{F}$) and conjugates of distinct components (in the same component of $\mathcal{F}$) have trivial intersections. The ψ-equivariance of homothety h implies $\mathcal{G}[\mathcal{Y}_\tau]$ is $[\psi]$-invariant. By Gaboriau–Levitt's index theory, the complexity of $\mathcal{G}[\mathcal{Y}_\tau]$ is strictly less than that of $\mathcal{F}$ [GL95, Theorem III.2]. In particular, $\mathcal{G}[\mathcal{Y}_\tau]$ has finite type: P is finite and each component G_p is finitely generated. The "restriction" of ψ to $\mathcal{G}[\mathcal{Y}_\tau]$ determines a unique outer automorphism of the system.

We now characterize the elliptic/loxodromic elements in $\mathcal{F}$ in the limit forest $(\mathcal{Y}_\tau, d_\infty)$:

Proposition II.2 (cf. [BFH97, Proposition 1.6])**.** *Let $\psi\colon \mathcal{F} \to \mathcal{F}$ be an automorphism, $\tau\colon \mathcal{T} \to \mathcal{T}$ an expanding irreducible train track for ψ, and $(\mathcal{Y}_\tau, d_\infty)$ the limit forest for $[\tau]$ with the constructed equivariant metric PL-map $\pi\colon (\mathcal{T}, d_\tau) \to (\mathcal{Y}_\tau, d_\infty)$.*

If $x \in \mathcal{F}$ is a $\mathcal{T}$-loxodromic element, then the following statements are equivalent:

1. *the element x is $\mathcal{Y}_\tau$-loxodromic;*
2. *the element x $[\psi]$-grows exponentially rel. d_τ: $\lim\limits_{m\to\infty} \frac{1}{m} \log \|\psi^m(x)\|_{d_\tau} = \log \lambda[\tau]$; and*
3. *the $\mathcal{T}$-axis for $\psi^m(x)$ has an arbitrarily long leaf segment of $\mathcal{L}^+[\tau]$ for $m \gg 1$.*

The restriction of ψ to the $[\psi]$-invariant subgroup system $\mathcal{G}[\mathcal{Y}_\tau]$ of $\mathcal{Y}_\tau$-point stabilizers has constant growth rel. d_τ: $\{\|\psi^m(x)\|_{d_\tau} : m \geq 0\}$ is bounded for all $x \in \mathcal{G}[\mathcal{Y}_\tau]$.

Proof. Let $\lambda \coloneqq \lambda[\tau] > 1$, $C \coloneqq C[\tau]$ a cancellation constant for $\tau\colon (\mathcal{T}, d_\tau) \to (\mathcal{T}, d_\tau)$, and $C' \coloneqq \frac{C}{\lambda-1}$ a cancellation constant for $\pi\colon (\mathcal{T}, d_\tau) \to (\mathcal{Y}_\tau, d_\infty)$. Fix a line l in $\mathcal{T}$.

Case 1: $d_\infty(\pi(p), \pi(q)) > 2C' + L$ for some $k \geq 0$, $p, q \in \tau_*^k(l)$, and $L > 0$. By definition of d_∞ (construction of π), $d_\tau(\tau^m(p), \tau^m(q)) > \lambda^m(2C' + L)$ for $m \gg 1$. Pick $m \gg 1$ and $r_m, s_m \in [\tau^m(p), \tau^m(q)]$ so that $d_\tau(\tau^m(p), r_m), d_\tau(s_m, \tau^m(q)) > \lambda^m C'$ and $d_\tau(r_m, s_m) > \lambda^m L$. By bounded cancellation (for τ^m), the interval $I_m \coloneqq [r_m, s_m]$ is disjoint from $\tau^m(\tau_*^k(l) \setminus [p, q])$ in (T, d_τ). So I_m is an interval in $\tau_*^{m+k}(l)$.

Let $N \coloneqq N(p, q)$ be the number of vertices in interval (p, q). Then I_m is covered by $N + 1$ leaf segments (of $\mathcal{L}^+[\tau]$) as τ is a train track. By the pigeonhole principle, I_m

(and hence $\tau_*^{m+k}(l)$) contains a leaf segment with d_τ-length $> \frac{\lambda^m L}{N+1}$; therefore, the line $\tau_*^n(l)$ in $\mathcal{T}$ contains arbitrarily long leaf segments for $m \gg 1$.

<u>Case 2:</u> $\pi(\tau_*^m(l))$ has diameter $\leq 2C'$ for all $m \geq 0$. We claim that any leaf segment in the line $\tau_*^m(l)$ $(m \geq 0)$ has d_τ-length $\leq 2C'$. For the contrapositive, suppose some $\tau_*^m(l)$ has a leaf segment with d_τ-length $L > 2C'$. By the train track property and bounded cancellation, $\tau_*^{m+1}(l)$ has a leaf segment with d_τ-length $\geq \lambda L - 2C > L$. By induction, $\pi(\tau_*^{m+m'}(l))$ has diameter $\geq \lambda^{m'}(L - 2C')$ for $m' \geq 0$ and $\lambda^{m'}(L-2C') > 2C'$ for $m' \gg 1$.

We finally return to the proof of the proposition. Fix a $\mathcal{T}$-loxodromic element $x \in \mathcal{F}$ and let $l \subset \mathcal{T}$ be its axis; in particular, $\pi(l)$ is x-invariant by equivariance of π. As τ is λ-Lipschitz with respect to d_τ, $\limsup_{m\to\infty} \frac{1}{m} \log \|\psi^m(x)\|_{d_\tau} \leq \log \lambda$.

Case–i: $d_\infty(\pi(p), \pi(q)) > 2C'$ for some $k \geq 0$ and $p, q \in \tau_*^k(l)$. The line $\tau_*^m(l)$, the axis for $\phi^m(x)$ in $\mathcal{T}$, contains an arbitrarily long leaf segment for $m \gg 1$ by the Case 1 analysis. By bounded cancellation (for π), some nondegenerate interval I in $[\pi(p), \pi(q)]_\infty$ is disjoint from $\pi(\tau_*^k(l) \setminus [p,q])$. Since $\tau_*^k(l)$ is the axis for $\psi^k(x)$, it contains disjoint translates $[p,q]$, $\psi^k(x^{-n}) \cdot [p,q]$, $\psi^k(x^n) \cdot [p,q]$ for some $n \gg 1$. Then $\psi^k(x^{-n}) \cdot I$ and $\psi^k(x^n) \cdot I$ are in distinct components of $\mathcal{Y}_\tau \setminus I$ and $\psi^k(x)$ is $\mathcal{Y}_\tau$-loxodromic. Since $\|\cdot\|_{d_\infty} \leq \|\cdot\|_{d_\tau}$ and $\|\psi(\cdot)\|_{d_\infty} = \lambda \|\cdot\|_{d_\infty}$, we get $\log \lambda \leq \liminf_{m\to\infty} \frac{1}{m} \log \|\psi^m(x)\|_{d_\tau}$ and x is $\mathcal{Y}_\tau$-loxodromic

Case–ii: $\pi(\tau_*^m(l))$ has diameter $\leq 2C'$ for all $m \geq 0$. Any leaf segment in $\tau_*^m(l)$ $(m \geq 0)$ have d_τ-length $\leq 2C'$ by Case 2 analysis. Let N be the number of vertices in a fundamental domain of (x acting on) l. By the train track property, the fundamental domain of $\tau_*^m(l)$ is covered by $N+1$ leaf segments and $\|\psi^m(x)\|_{d_\tau} \leq 2C'(N+1)$. But x acts on $\mathcal{Y}_\tau$ by an isometry and $\pi(l) \subset \mathcal{Y}_\tau$ is x-invariant, so x must be $\mathcal{Y}_\tau$-elliptic. □

We now introduce the notion *factored* $\mathcal{F}$-forests. Suppose the stable laminations $\mathcal{L}^+[\tau]$ have components Λ_i^+ $(1 \leq i \leq k)$. The $\mathcal{F}$-orbits of edges in $\mathcal{T}$ can be partitioned into *blocks* $\mathcal{B}_i$ indexed by the components $\Lambda_i^+ \subset \mathcal{L}^+[\tau]$. For $p, q \in \mathcal{T}$, let $d_\tau^{(i)}$ be the d_τ-length of intersection of the interval $[p,q]$ and the subforest spanned by $\mathcal{B}_i$; this defines an $\mathcal{F}$-invariant convex pseudometric $d_\tau^{(i)}$ on $\mathcal{T}$. The metric d_τ is a sum of the pseudometrics $d_\tau^{(i)}$, denoted $\oplus_{i=1}^k d_\tau^{(i)}$; we call $\oplus_{i=1}^k d_\tau^{(i)}$ a <u>factored</u> $\mathcal{F}$-invariant convex metric and $(\mathcal{T}, \oplus_{i=1}^k d_\tau^{(i)})$ a <u>factored</u> $\mathcal{F}$-forest. By τ^k-iteration, the metric d_∞ is a sum of pseudometrics $d_\infty^{(i)}$ and we get a factored limit forest $(\mathcal{Y}_\tau, \oplus_{i=1}^k d_\infty^{(i)})$ for $[\tau^k]$ and the ψ^k-equivariant pretree-automorphism $h^k \colon \mathcal{Y}_\tau \to \mathcal{Y}_\tau$ is a λ^k-homothety with respect to each *factor* $d_\infty^{(i)}$. The next lemma is the cornerstone of our universality result:

Lemma V.3 (cf. [BFH97, Lemma 3.4])**.** *Let $\psi \colon \mathcal{F} \to \mathcal{F}$ be an automorphism, $\tau \colon \mathcal{T} \to \mathcal{T}$ an expanding irreducible train track for ψ, $(\mathcal{Y}_\tau, d_\infty)$ the limit forest for $[\tau]$, and $\lambda \coloneqq \lambda[\tau]$.*

If $(\mathcal{T}, d_\tau) \to (\mathcal{Y}, \delta)$ is an equivariant PL-map and the k-component lamination $\mathcal{L}^+[\tau]$ is in $\mathbb{R}(\mathcal{Y}, \delta) \subset \mathbb{R}(\mathcal{T})$, then the sequence $(\mathcal{Y}\psi^{mk}, \lambda^{-mk}\delta)_{m \geq 0}$ converges to $(\mathcal{Y}_\tau, \oplus_{i=1}^k c_i\, d_\infty^{(i)})$, where $d_\infty = \oplus_{i=1}^k d_\infty^{(i)}$ and $c_i > 0$.

Remark. Factored $\mathcal{F}$-forests are needed for this lemma when $k \geq 2$; similarly, the sequence $(\mathcal{Y}\psi^m, \lambda^{-m}\delta)_{m\geq 0}$ will not converge in general (but is asymptotically periodic) when $k \geq 2$.

We give the proof in Section V.1. In particular, if $\tau'\colon \mathcal{T}' \to \mathcal{T}'$ is another expanding irreducible train track for ψ and $\mathcal{F}[\mathcal{T}'] = \mathcal{F}[\mathcal{T}]$, then the limit forest for $[\tau']$ is equivariantly homothetic to $(\mathcal{Y}_\tau, d_\infty)$ — set $(\mathcal{Y}, \delta) \coloneqq (\mathcal{T}', d_{\tau'})$, apply the lemma, then observe that the sequence $(c_i)_{i=1}^k$ must be constant in this case. A minimal $\mathcal{F}$-forest $(\mathcal{Y}, \delta)$ with trivial arc stabilizers is an expanding forest for $[\psi]$ like $\mathcal{Y}_\tau$ if there is:

1. a ψ-equivariant expanding homothety $(\mathcal{Y}, \delta) \to (\mathcal{Y}, \delta)$; and
2. an equivariant PL-map $(\mathcal{T}, d_\tau) \to (\mathcal{Y}, \delta)$.

Corollary II.3. *Let $\psi\colon \mathcal{F} \to \mathcal{F}$ be an automorphism, $\tau\colon \mathcal{T} \to \mathcal{T}$ an expanding irreducible train track for ψ, and $(\mathcal{Y}_\tau, d_\infty)$ the limit forest for $[\tau]$. Any expanding forests for $[\psi]$ like $\mathcal{Y}_\tau$ is uniquely equivariantly homothetic to $(\mathcal{Y}_\tau, d_\infty)$.*

Proof. Let $(\mathcal{Y}, \delta)$ be an expanding forest for $[\psi]$ like $\mathcal{Y}_\tau$, $f\colon (\mathcal{T}, d_\tau) \to (\mathcal{Y}, \delta)$ an equivariant PL-map, $g\colon (\mathcal{Y}, \delta) \to (\mathcal{Y}, \delta)$ the ψ-equivariant expanding s-homothety, $x \in \mathcal{F}$ a $\mathcal{Y}$-loxodromic element. By equivariance of f, the element x is $\mathcal{T}$-loxodromic as well with axis $l_x \subset \mathcal{T}$. Let $[p_0, x \cdot p_0] \subset l_x$ be (the closure of) a fundamental domain of x acting on l_x. The interval $[p_0, x \cdot p_0]$ is concatenation of $k \geq 1$ leaf segments (of $\mathcal{L}^+[\tau]$). Choose $m \gg 1$ so that $\|\psi^m(x)\|_\delta = s^m \|x\|_\delta > k\, 2C[f]$. Note that the action of $\psi^m(x)$ on its axis has a fundamental domain $[p_m, \psi^m(x) \cdot p_m]$ covered by k leaf segments as τ is a train track. So $\delta(f(p_m), f(\psi^m(x) \cdot p_m)) > k\, 2C[f]$ and, by the pigeonhole principle, $[p_m, \psi^m(x) \cdot p_m]$ contains a leaf segment $[q, r]$ with $\delta(f(q), f(r)) > 2C[f]$.

Let $l \supset [q, r]$ represent some leaf $[l] \in \mathcal{L}^+[\tau]$. Bounded cancellation implies the components of $l \setminus [q, r]$ have f-images with disjoint closures. By quasiperiodicity of $[l]$ and equivariance of f, both ends of l have unbounded f-images, i.e. $[l] \in \mathrm{dom}(f_*) = \mathbb{R}(\mathcal{Y}, \delta)$ (Corollary I.4, Claim I.5). Finally, the closure of $[l]$ in $\mathbb{R}(\mathcal{T})$, a component $\Lambda_i^+ \subset \mathcal{L}^+[\tau]$, is a subset of $\mathbb{R}(\mathcal{Y}, \delta)$ by quasiperiodicity of $[l]$. Note that the ψ-equivariant homothety g induces a homeomorphism $g_*\colon \mathbb{R}(\mathcal{Y}, \delta) \to \mathbb{R}(\mathcal{Y}, \delta)$ that is restriction of the homeomorphism $\tau_*\colon \mathbb{R}(\mathcal{T}) \to \mathbb{R}(\mathcal{T})$. So $\mathcal{L}^+[\tau] \subset \mathbb{R}(\mathcal{Y}, \delta)$ since τ_* acts transitively on the k components of $\mathcal{L}^+[\tau]$. Set $\lambda \coloneqq \lambda[\tau]$; by Lemma V.3, the sequence $(\mathcal{Y}\psi^{mk}, \lambda^{-mk}\delta)_{m\geq 0}$ converges to the factored $\mathcal{F}$-forest $(\mathcal{Y}_\tau, \oplus_{i=i}^k c_i\, d_\infty^{(i)})$ for some $c_i > 0$. Yet $(\mathcal{Y}, \delta)$ is equivariantly isometric to $(\mathcal{Y}\psi, s^{-1}\delta)$; thus $s = \lambda$, $c_i = c_{i+1}$ $(i < k)$, $(\mathcal{Y}, \delta)$ is equivariantly isometric to $(\mathcal{Y}_\tau, c_1\, d_\infty)$, and the equivariant isometry is unique [CM87, Theorem 3.7]. □

II.2.3 Iterated turns

We have already shown how iterating an edge in $\mathcal{T}$ by the train track τ produces the system of stable laminations $\mathcal{L}^+[\tau]$. Later, we will consider how $\mathcal{L}^+[\tau]$ determines laminations in a free splitting of the subgroup system $\mathcal{G}[\mathcal{Y}_\tau]$.

Let $\mathcal{T}'$ be a free splitting of $\mathcal{F}$ whose free factor system $\mathcal{F}[\mathcal{T}']$ is trivial. Then there is an equivariant PL-map $f\colon (\mathcal{T}', d') \to (\mathcal{T}, d_\tau)$. Let γ be a line in $(\mathcal{Y}_\tau, d_\infty)$, $\pi^*(\gamma)$ its lift to $(\mathcal{T}, d_\tau)$, and $f^*(\pi^*(\gamma))$ its lift to $(\mathcal{T}', d')$. Denote the ends of γ by ε_i $(i = 1, 2)$. Let $T \subset \mathcal{T}$ be the component containing $\pi^*(\gamma)$ and $T' \subset \mathcal{T}'$, $Y_\tau \subset \mathcal{Y}_\tau$, and $F \subset \mathcal{F}$ the matching components. Denote the first return maps of τ, h, and ψ on T, Y_τ, and F by $\tilde{\tau}$, $\tilde{h}$, and φ respectively. For the rest of the section, redefine λ to be the stretch factor of the expanding homothety $\tilde{h}$.

Suppose $\circ$ is a point on the line γ with a nontrivial stabilizers $G_\circ := \mathrm{Stab}_F(\circ)$. Let d_i $(i = 1, 2)$ be the direction at $\circ$ containing ε_i. By Gaboriau–Levitt index theory, $\tilde{h}^j(\circ) = y \cdot \circ$ and $\tilde{h}^j(d_i) = ys_i \cdot d_i$ for some $y \in F$, $s_i \in G_\circ$, and minimal $j \geq 1$. Since F acts on Y_τ with trivial arc stabilizers, the elements ys_1, ys_2 are unique and $s_1^{-1}s_2 \in G_\circ$ is independent of the chosen $y \in F$.

Set $y_0 := \epsilon$ to be the trivial element and $y_{m+1} := \varphi^{mj}(ys_1)y_m$ for $m \geq 0$. Let $T'(G)$ be the characteristic convex subset of T' for a nontrivial subgroup $G \leq F$. Since T' is simplicial, the characteristic convex subset $T'(G)$ is *closed* and we have the *closest point retraction* $T' \to T'(G)$; it extends uniquely to the *ends-completions*. Let $q'_{i,m}$ be the closest point projection of $f^*(\pi^*(\tilde{h}_*^{mj}(\varepsilon_i)))$ to $T'(\varphi^{mj}(G_\circ))$. Set $\tau_\circ := (ys_1)^{-1} \cdot \tilde{\tau}^j$ and $h_\circ := (ys_1)^{-1} \cdot \tilde{h}^j$ to be $\psi_\circ$-equivariant maps for an automorphism $\psi_\circ\colon F \to F$ in the outer class $[\varphi^j]$. As $h_\circ$ fixes $\circ$, we get $\psi_\circ(G_\circ) = G_\circ$ and $y_m^{-1} \cdot T'(\varphi^{mj}(G_\circ))$ is the characteristic convex subset for $\psi_\circ^m(G_\circ) = G_\circ$. Thus $q_{i,m} := y_m^{-1} \cdot q'_{i,m}$ is in $T'(G_\circ)$ for $i = 1, 2$ and $m \geq 0$. The interval $[q_{1,m}, q_{2,m}]$ in $T'(G_\circ)$, i.e. the closest point projection of $f^*(\pi^*(h_\circ^m(\gamma)))$, is the <u>turn</u> in $f^*(\pi^*(h_\circ^m(\gamma)))$ about $T'(G_\circ)$.

Since $h_\circ(d_1) = d_1$, the ends $h_{\circ *}^m(\varepsilon_1)$ $(m \geq 0)$ are in fact ends of d_1. If $h_{\circ *}(\varepsilon_1) = \varepsilon_1$, then the sequence $(q_{1,m})_{m \geq 0}$ is constant. Otherwise, the ends $h_{\circ *}^m(\varepsilon_1)$ are distinct for $m \geq 0$. Let $\gamma_{1,m}$ be the line in d_1 determined by $h_{\circ *}^{m+1}(\varepsilon_1)$ and $h_{\circ *}^m(\varepsilon_1)$. As $h_\circ$ is an expanding homothety, the distance $d_\infty(\circ, \gamma_{1,m}) > 0$ from $\circ$ to $\gamma_{1,m}$ grows exponentially in m. So $d_\infty(\circ, \gamma_{1,M_1}) > 2C[\pi \circ f]$ for some minimal $M_1 \geq 0$ and the line $f^*(\pi^*(\gamma_{1,m}))$ is disjoint from $T'(G_\circ)$ for $m \geq M_1$ by bounded cancellation (see Figure 2). In particular, the ends $f^*(\pi^*(h_{\circ *}^{m+1}(\varepsilon_1)))$ and $f^*(\pi^*(h_{\circ *}^m(\varepsilon_1)))$ have the same closest point projection to $T'(G_\circ)$ and the sequence $(q_{1,m})_{m \geq M_1}$ is constant.

Since $h_\circ(d_2) = s_1^{-1}s_2 \cdot d_2$, the ends $f^*(\pi^*(h_{\circ *}^{m+1}(\varepsilon_2)))$ and $\psi_\circ^m(s_1^{-1}s_2) \cdot f^*(\pi^*(h_{\circ *}^m(\varepsilon_2)))$ have the same closest point projection to $T'(G_\circ)$ for $m \gg 1$ by similar bounded cancellation reasoning, i.e. $q_{2,m+1} = \psi_\circ^m(s_1^{-1}s_2) \cdot q_{2,m}$ for some minimal $M_2 \geq 0$ and all $m \geq M_2$.

Set $M = \max(M_1, M_2)$. The sequence $[q_{1,M+m}, q_{2,M+m}]_{m \geq 0}$ of intervals is well-defined for the line γ and point $\circ \in \gamma$ as M_1 and M_2 were chosen minimally. An *iterated turn over* $T'(G_\circ)$ *rel.* $\psi_\circ|_{G_\circ}$ is any such sequence of intervals. More generally, we define an <u>iterated turn over T rel. φ</u>: pick arbitrary points $p_i \in T$ $(i = 1, 2)$ and elements $x_i \in F$; set $p_{i,0} := p_i$ and $p_{i,m+1} := \varphi(x_i) \cdot p_{i,m}$ for $m \geq 0$; the sequence $[p_{1,m}, p_{2,m}]_{m \geq 0}$ is the iterated turn denoted by $(p_1, p_2 : x_1, x_2; \varphi)_T$. Any iterated turn $(p_1, p_2 : x_1, x_2; \varphi)_T$ translates to a unique *normal* form $(p_1, p_2 : \epsilon, x_1^{-1}x_2; \tilde{\varphi})_T$ with $\tilde{\varphi}\colon y \mapsto x_1^{-1}\varphi(y)x_1$.

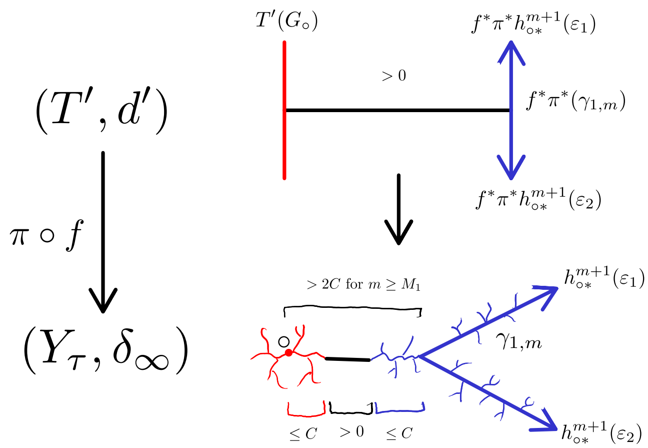

Figure 2: For $m \geq M_1$, the line $f^*(\pi^*(\gamma_{1,m}))$ cannot intersect $T'(G_\circ)$.

The next proposition characterizes the growth of an iterated turn over T rel. φ:

Proposition II.4. *Let $\psi\colon \mathcal{F} \to \mathcal{F}$ be an automorphism, $\tau\colon \mathcal{T} \to \mathcal{T}$ an expanding irreducible train track for ψ, and $(\mathcal{Y}_\tau, d_\infty)$ the limit forest for $[\tau]$ with the constructed equivariant metric PL-map $\pi\colon (\mathcal{T}, d_\tau) \to (\mathcal{Y}_\tau, d_\infty)$. Choose a nondegenerate component $T \subset \mathcal{T}$, corresponding components $F \subset \mathcal{F}$, $Y_\tau \subset \mathcal{Y}_\tau$, and a positive iterate ψ^k that preserves F. Let $\tilde{h}\colon (Y_\tau, d_\infty) \to (Y_\tau, d_\infty)$ be the φ-equivariant λ-homothety, where φ is in the outer automorphism $[\psi^k|_F]$ and $\lambda \coloneqq (\lambda[\tau])^k$. Finally, for $i = 1, 2$, pick $p_i \in T$ and $x_i \in F$.*

The point $p_{i,m} \coloneqq \varphi^{-1}(x_i) \cdots \varphi^{-m}(x_i) \cdot p_i$ in $(T\varphi^m, \lambda^{-m} d_\tau)$ converges to $\star_i$ in $(\overline{Y}_\tau, d_\infty)$ as $m \to \infty$, where $\star_i$ is the unique fixed point of $x_i^{-1} \cdot \tilde{h}$ in the metric completion $(\overline{Y}_\tau, d_\infty)$; concretely: $\lim_{m\to\infty} \lambda^{-m} d_\tau(p_{1,m}, p_{2,m}) = \lim_{m\to\infty} \lambda^{-m} d_\infty(\pi(p_{1,m}), \pi(p_{2,m})) = d_\infty(\star_1, \star_2)$.

If $x_1^{-1}x_2$ fixes $\star_1$, then $\star_1 = \star_2$ and the m^{th} term $[p_{1,m}, p_{2,m}]$ of the iterated turn $(p_1, p_2 : x_1, x_2; \varphi)_T$ has d_τ-length $\leq (m+1)A$ for some constant $A \geq 1$. Otherwise, $\star_1 \neq \star_2$ and the iterated turn has arbitrarily long leaf segments of $\mathcal{L}^+[\tau]$.

The limit $[\star_1, \star_2] \subset \overline{Y}_\tau$ of an iterated turn is independent of the points $p_1, p_2 \in T$. Thus we introduce the notion of an <u>*algebraic* iterated turn over F rel. φ</u>, denoted $(x_1, x_2; \varphi)_F$.

Proof. Let $p_1, p_2 \in T$ and $x_1, x_2 \in F$. For $i = 1, 2$, set $p_{i,0} \coloneqq p_i$ and $p_{i,m+1} \coloneqq \varphi^m(x_i) \cdot p_{i,m}$ for $m \geq 0$. Recall that T and $T\varphi^m$ are the same pretrees but with different actions; thus, in $T\varphi^m$, we have $p_{i,m} = \varphi^{-1}(x_i) \cdots \varphi^{-m}(x_i) \cdot p_i$ for $m \geq 0$. As $\pi\colon (T, d_\tau) \to (Y_\tau, d_\infty)$ is an equivariant metric PL-map, so is the composition

$$\pi_m\colon (T\varphi^m, \lambda^{-m} d_\tau) \xrightarrow{\pi} (Y_\tau\varphi^m, \lambda^{-m} d_\infty) \xrightarrow{\tilde{h}^{-m}} (Y_\tau, d_\infty).$$

The point p_i in (T, d_τ) projects (via π) to $\pi(p_i)$ in (Y_τ, d_∞); the point $p_{i,m}$ in $(T\varphi^m, \lambda^{-m} d_\tau)$ projects (via π_m) to

$$\begin{aligned} \pi_m(p_{i,m}) \coloneqq \tilde{h}^{-m}(\pi(p_{i,m})) &= \varphi^{-1}(x_i) \cdots \varphi^{-m}(x_i) \cdot \tilde{h}^{-m}(\pi(p_i)) \qquad (\, p_{i,m}, p_i \in T\varphi^m \,) \\ &= (x_i^{-1} \cdot \tilde{h})^{-m}(\pi(p_i)) \qquad (\, p_i \in T \,) \end{aligned}$$

in (Y_τ, d_∞) for $m \geq 1$ — in the last line, $x_i^{-1} \cdot \tilde{h}$ is a λ-homothety $(Y_\tau, d_\infty) \to (Y_\tau, d_\infty)$. Since $(x_i^{-1} \cdot \tilde{h})^{-1}$ is contracting, the point $\pi_m(p_{i,m})$ converges (as $n \to \infty$) to the unique fixed point $\star_i$ of $(x_i^{-1} \cdot \tilde{h})^{-1}$ (and $x_i^{-1} \cdot \tilde{h}$) in the metric completion $(\overline{Y}_\tau, d_\infty)$ by the contraction mapping theorem; note that $x_1^{-1} x_2 \cdot \star_1 = \star_1$ if and only if $\star_1 = \star_2$. Thus the (π_m-projection of the) point $p_{i,m}$ in $(T\varphi^m, \lambda^{-m} d_\tau)$ converges (as $m \to \infty$) to $\star_i$ in $(\overline{Y}_\tau, d_\infty)$; in particular,

$$\lim_{m \to \infty} \lambda^{-m} d_\infty(\pi(p_{1,m}), \pi(p_{2,m})) = \lim_{m \to \infty} d_\infty(\pi_m(p_{1,m}), \pi_m(p_{2,m})) = d_\infty(\star_1, \star_2).$$

Let $\tilde{\tau} \colon T \to T$ be the φ-equivariant translate of a component of τ^k. The interval $[p_{1,m}, p_{2,m}] \subset T$, the m^{th} term in $(p_1, p_2 : x_1, x_2; \varphi)_T$, is covered by these $2m+1$ intervals:

$$\begin{aligned} &\varphi^{m-1}(x_1) \cdots \varphi(x_1) \cdot [x_1 \cdot p_1, \tilde{\tau}(p_1)], \ldots,\ \varphi^{m-1}(x_1) \cdot [\tilde{\tau}^{m-2}(x_1 \cdot p_1), \tilde{\tau}^{m-1}(p_1)], \\ &[\tilde{\tau}^{m-1}(x_1 \cdot p_1), \tilde{\tau}^m(p_1)],\ [\tilde{\tau}^m(p_1), \tilde{\tau}^m(p_2)],\ [\tilde{\tau}^m(p_2), \tilde{\tau}^{m-1}(x_2 \cdot p_2)], \\ &\varphi^{m-1}(x_2) \cdot [\tilde{\tau}^{m-1}(p_2), \tilde{\tau}^{m-2}(x_2 \cdot p_2)], \ldots,\ \varphi^{m-1}(x_2) \cdots \varphi(x_2) \cdot [\tilde{\tau}(p_2), x_2 \cdot p_2]. \end{aligned}$$

Set $D \coloneqq \max\{d_\tau(x_i \cdot p_i, \tilde{\tau}(p_i) : i = 1, 2\}$ and $D' \coloneqq \frac{D}{\lambda - 1}$.

Recall that $\lim_{m' \to \infty} \lambda^{-m'} d_\tau(\tilde{\tau}^{m'}(p_{1,m}), \tilde{\tau}^{m'}(p_{2,m})) = d_\infty(\pi(p_{1,m}), \pi(p_{2,m}))$. For $m' \geq 0$, we get a similar covering of $[p_{1,m+m'}, p_{2,m+m'}]$ by $2m'+1$ intervals with the "middle" $[\tilde{\tau}^{m'}(p_{1,m}), \tilde{\tau}^{m'}(p_{2,m})]$. Since $\tilde{\tau}$ is λ-Lipschitz with respect to d_τ, the sum of the d_τ-lengths of all intervals but the middle in this covering is $\leq \lambda^{m'} 2D'$. By the triangle inequality,

$$\lambda^{-(m+m')} \left| d_\tau(p_{1,m+m'}, p_{2,m+m'}) - d_\tau(\tilde{\tau}^{m'}(p_{1,m}), \tilde{\tau}^{m'}(p_{2,m})) \right| \leq \lambda^{-m} 2D'.$$

Fix $\epsilon > 0$; then $\lambda^{-m} 2D' < \epsilon$ and $|\lambda^{-m} d_\infty(\pi(p_{1,m}), \pi(p_{2,m})) - d_\infty(\star_1, \star_2)| < \epsilon$ for some $m \gg 1$. Similarly, $\lambda^{-m} \left| \lambda^{-m'} d_\tau(\tilde{\tau}^{m'}(p_{1,m}), \tilde{\tau}^{m'}(p_{2,m})) - d_\infty(\pi(p_{1,m}), \pi(p_{2,m})) \right| < \epsilon$ and $\left| \lambda^{-(m+m')} d_\tau(p_{1,m+m'}, p_{2,m+m'}) - d_\infty(\star_1, \star_2) \right| < 3\epsilon$ for $m' \gg 1$, i.e.

$$\lim_{m \to \infty} \lambda^{-m} d_\tau(p_{1,m}, p_{2,m}) = d_\infty(\star_1, \star_2).$$

Let $N(u, v)$ be the number of vertices in an interval $(u, v) \subset T$; set N to be the maximum of $N(p_1, p_2)$, $N(x_1 \cdot p_1, \tilde{\tau}(p_1))$, and $N(\tilde{\tau}(p_2), x_2 \cdot p_2)$. As $\tilde{\tau}$ is a train track, the interval $[p_{1,m}, p_{2,m}]$ is covered by $(2m+1)(N+1)$ leaf segments.

Suppose $\star_1 = \star_2$. We claim that any leaf segment (of $\mathcal{L}^+[\tau]$) in $[p_{1,m}, p_{2,m}]$ ($m \geq 0$) has uniformly bounded d_τ-length — this implies $[p_{1,m}, p_{2,m}]$ has d_τ-length $\leq (2m+1)(N+1)B$

for some bounding constant $B \geq 1$. We mimic Case 2 from the proof of Proposition II.2. For the contrapositive, suppose some term $[p_{1,m}, p_{2,m}]$ has a leaf segment with d_τ-length $L > 2(C[\pi] + D')$. By the train track property, bounded cancellation, and interval covering, $[p_{1,m+m'}, p_{2,m+m'}]$ has a leaf segment with d_τ-length $\geq \lambda^{m'}(L - 2C[\pi] - 2D')$ for $m' \geq 0$; in $(T\varphi^{m+m'}, \lambda^{-(m+m')}d_\tau)$, $[p_{1,m+m'}, p_{2,m+m'}]$ has length $\geq \lambda^{-m}(L - 2C[\pi] - 2D')$. In the limit (as $m' \to \infty$), $d_\infty(\star_1, \star_2) \geq \lambda^{-m}(L - 2C[\pi] - 2D') > 0$.

Suppose $\star_1 \neq \star_2$. Set $L := \frac{1}{2}d_\infty(\star_1, \star_2) > 0$; then $\lambda^{-m}d_\tau(p_{1,m}, p_{2,m}) > L$ for some $m \gg 1$. By the pigeonhole principle, the interval $[p_{1,m}, p_{2,m}]$ has a leaf segment with d_τ-length $\frac{\lambda^m L}{(2m+1)(N+1)}$, which can be arbitrarily large (in m). □

II.2.4 Nested iterated turns

The first part of the previous subsection explains how a line in $(\mathcal{Y}_\tau, d_\infty)$ determines algebraic iterated turns over $\mathcal{G}[\mathcal{Y}_\tau]$. We now give a similar discussion for an iterated turn over $\mathcal{T}'$.

Pick points $p'_1, p'_2 \in T'$ and elements $x_1, x_2 \in F$. Set $T'_m := T'\varphi^m$, $T_m := T\varphi^m$, $p'_{i,0} := p'_i$, $p'_{i,m} := \varphi^{-1}(x_i) \cdots \varphi^{-m}(x_i) \cdot p'_i$ in T'_m, and $p_{i,m} = f(p'_{i,m})$ for $m \geq 1$ and $i = 1, 2$. By Proposition II.4, the point $p_{i,m}$ in $(T_m, \lambda^{-m}d_\tau)$ converges (as $m \to \infty$) to $\star_i$ in $(\overline{Y}_\tau, d_\infty)$, the unique fixed point of $x_i^{-1} \cdot \tilde{h}$ in the metric completion. The λ-homothety $h_i := x_i^{-1} \cdot \tilde{h}$ is φ_i-equivariant for an automorphism $\varphi_i \colon F \to F$ in the outer class $[\varphi]$.

Case–a: $s := x_1^{-1}x_2 \in G_1 := \mathrm{Stab}_F(\star_1)$. Suppose G_1 is not trivial and let $a_{i,m}$ be the closest point projection of $p'_{i,m}$ to $T'(\varphi^m(G_1))$ for $m \geq 0$. As $\tilde{h}(\star_1) = x_1 \cdot \star_1$ and $\tilde{h}$ is φ-equivariant, we get $T'(\varphi^{m+1}(G_1)) = \varphi^m(x_1) \cdot T'(\varphi^m(G_1))$, $a_{1,m+1} = \varphi^m(x_1) \cdot a_{1,m}$, and

$$a_{2,m+1} = \varphi^m(x_1)\varphi^m(s) \cdot a_{2,m} = \varphi^m(x_1) \cdots \varphi(x_1)x_1\varphi_1^m(s) \cdots \varphi_1^m(s)s \cdot a_{2,0} \quad \text{for } m \geq 0.$$

Thus the closest point projection to $T'(\varphi^m(G_1))$ of the m^{th} term of the given iterated turn $(p'_1, p'_2 : x_1, x_2; \varphi)_{T'}$ is a translate of the m^{th} term in $(a_{1,0}, a_{2,0} : \epsilon, s; \varphi_1|_{G_1})_{T'(G_1)}$, where $m \geq 0$ and ϵ is the identity element. Hence we have an algebraic iterated turn $(\epsilon, s; \varphi_1|_{G_1})_{G_1}$ that is well-defined for the algebraic iterated turn $(x_1, x_2; \varphi)_F$.

Case–b: $\star_1 \neq \star_2$. Suppose G_1 is not trivial — the argument is symmetric if $\mathrm{Stab}_F(\star_2)$ is not trivial — and let d be the direction at $\star_1$ containing $\star_2$. By Gaboriau–Levitt index theory, $h_1^j(d) = t \cdot d$ for some $t \in G_1$ and minimal $j \geq 1$. For $m \gg 1$, $\pi_m(p_{2,n}) = h_2^{-m}(\pi(p_2))$ is in the direction d since $h_2^{-m}(\pi(p_2)) \to \star_2$ in $(\overline{Y}_\tau, d_\infty)$. For $m \gg 1$ and $m' \geq 0$, the interval $[p_{2,m+m'j}, \tilde{\tau}^{m'j}(p_{2,m})]$ in $(T_{m+m'j}, \lambda^{-m-m'j}d_\tau)$ is disjoint from $T_{m+m'j}(G_1)$ by bounded cancellation (see Figure 3); or equivalently, the interval $[p_{2,m+m'j}, \tilde{\tau}^{m'j}(p_{2,m})]$ in T_m is disjoint from $T_m(\varphi^{m'j}(G_1))$. In fact, the $\lambda^{-m}d_\tau$-distance in T_m from $[p_{2,m+m'j}, \tilde{\tau}^{m'j}(p_{2,m})]$ to $T_m(\varphi^{m'j}(G_1))$ can be arbitrarily large for $m' \gg 1$.

Set $z_0 := \epsilon$ and $z_{m'+1} := \varphi^{m'}(x_1)z_{m'}$. Let $b'_{i,m'}$ $(i = 1, 2)$ be the closest point projection of $p'_{i,m+m'j}$ to $T'_m(\varphi^{m'j}(G_1)) = z_{m'j} \cdot T'_m(G_1)$ and set $b_{i,m'} := z_{m'j}^{-1} \cdot b'_{i,m'}$ in $T'_m(G_1)$. Following

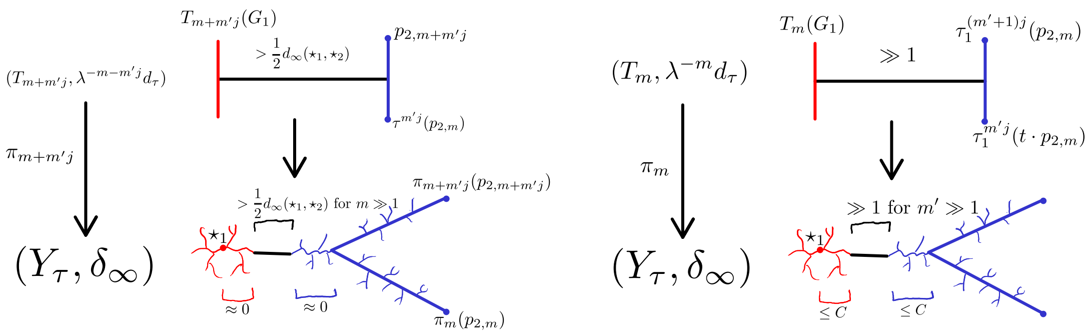

Figure 3: The two figures illustrating certain closest point projections are the same.

the definitions, $z_{m'j}^{-1} \cdot p'_{1,m+m'j} = p'_{1,m}$ in T'_m and $z_{m'j}^{-1} \cdot \tilde{\tau}^{m'j} = \tau_1^{m'j}$ in T_m, where $\tau_1 := x_1^{-1} \cdot \tilde{\tau}$; in particular, $b_{1,m'} = b_{1,0}$ for $m' \geq 0$. Since $h_1^j(d) = t \cdot d$, bounded cancellation implies the $\lambda^{-m} d_\tau$-distance in T_m from $[\tau_1^{(m'+1)j}(p_{2,m}), \varphi_1^{m'j}(t) \cdot \tau_1^{m'j}(p_{2,m})]$ to $T_m(G_1)$ is arbitrarily large for $m' \gg 1$ (see Figure 3).

So $[z_{(m'+1)j}^{-1} \cdot p_{2,m+(m'+1)j}, \varphi_1^{m'j}(t) z_{m'j}^{-1} \cdot p_{2,m+m'j}]$ is arbibtrarily far from $T_m(G_1)$ by transitivity. By bounded cancellation, $[z_{(m'+1)j}^{-1} \cdot p'_{2,m+(m'+1)j}, \varphi_1^{m'j}(t) z_{m'j}^{-1} \cdot p'_{2,m+m'j}]$ is disjoint from $T'_m(G_1)$ for $m' \gg 1$, i.e. $b_{2,m'+1} = \varphi_1^{m'j}(t) \cdot b_{2,m'}$ for $m' \gg 1$.

Thus, for some $M' \gg 1$, the sequence $[b_{1,M'+m'}, b_{2,M'+m'}]_{m' \geq 0}$ is an iterated turn over $T'_m(G_1)$ rel. $\varphi_1^j|_{G_1}$: $(b_{1,M'}, b_{2,M'} : \epsilon, t; \varphi_1^j|_{G_1})_{T'_m(G_1)}$. The corresponding algebraic iterated turn $(\epsilon, t; \varphi_1^j|_{G_1})_{G_1}$ is well-defined for $(x_1, x_2; \varphi)_F$.

Now suppose $\circ \in (\star_1, \star_2)$ has a nontrivial stabilizer $G_\circ := \mathrm{Stab}_F(\circ)$. Let d_i $(i = 1, 2)$ be the direction at $\circ$ containing $\star_i$. By index theory again, $h_1^l(\circ) = x \cdot \circ$ and $h_1^l(d_i) = x s_i \cdot d_i$ for some $x \in F$, $s_i \in G_\circ$, and minimal $l \geq 1$. Since F acts on Y_τ with trivial arc stabilizers, the elements xs_1, xs_2 are unique and $s_1^{-1} s_2 \in G_\circ$ is independent of the chosen $x \in F$. For $m \gg 1$, $\pi_m(p_{i,m})$ is in the direction d_i since $\pi_m(p_{i,m}) \to \star_i$. What follows can be proven with a variation of the bounded cancellation arguments used in the preceding paragraphs. For $m, m' \gg 1$, the interval $[p_{i,m+m'l}, \tilde{\tau}^{m'l}(p_{i,m})]$ in $(T_m, \lambda^{-m} d_\tau)$ is arbitrarily far from $T_m(\varphi^{m'l}(G_\circ))$.

Set $y_0 := \epsilon$, $y_{m'+1} := \varphi_\circ^{m'}(x) y_{m'}$, $\tau_\circ := x^{-1} \cdot \tau_1^l$, and $h_\circ := x^{-1} \cdot h_1^l$ to be $\varphi_\circ$-equivariant maps for an automorphism $\varphi_\circ \colon F \to F$ in the outer class $[\varphi_1^j]$. Let $c'_{i,m'} \in T'_m(\varphi^{m'l}(G_\circ))$ be the closest point projection of $p'_{i,m+m'l}$ and set $c''_{i,m'} := z_{m'l}^{-1} \cdot c'_{i,m'} \in T'_m(\varphi_1^{m'l}(G_\circ))$. Then $c_{i,m'} := y_{m'}^{-1} \cdot c''_{i,m'} \in T'_m(G_\circ)$ is the closest point projection of $y_{m'}^{-1} z_{m'l}^{-1} \cdot p'_{i,m+m'l}$. Since $h_\circ(d_i) = s_i \cdot d_i$, the interval $[\tau_\circ^{m'+1}(p_{i,m}), \varphi_\circ^{m'}(s_i) \cdot \tau_\circ^{m'}(p_{i,m})]$ is arbitrarily far from $T_m(G_\circ)$ for $m' \gg 1$. By transitivity, $[y_{m'+1}^{-1} z_{(m'+1)l}^{-1} \cdot p_{i,m+(m'+1)l}, \varphi_\circ^{m'}(s_i) y_{m'}^{-1} z_{m'}^{-1} \cdot p_{i,m+m'l}]$ is arbitrarily far from $T_m(G_\circ)$. As before, $[y_{m'+1}^{-1} z_{(m'+1)l}^{-1} \cdot p'_{i,m+(m'+1)l}, \varphi_\circ^{m'}(s_i) y_{m'}^{-1} z_{m'}^{-1} \cdot p'_{i,m+m'l}]$

is disjoint from $T'_m(G_\circ)$ for $m' \gg 1$, i.e. $c_{i,m'+1} = \varphi_\circ^{m'}(s_i) \cdot c_{i,m'}$ for $m' \gg 1$. Thus, for some $M'' \gg 1$, the sequence $[c_{1,M''+m'}, c_{2,M''+m'}]_{m' \geq 0}$ is an iterated turn over $T'_m(G_\circ)$ rel. $\varphi_\circ|_{G_\circ}$: $(c_{1,M''}, c_{2,M''} : s_1, s_2; \varphi_\circ|_{G_\circ})_{T'_m(G_\circ)}$. It is a "translate" of the normalized iterated turn: $(c_{1,M''}, c_{2,M''} : \epsilon, s_1^{-1}s_2; \varphi_\circ|_{G_\circ})_{T'_m(G_\circ)}$. The corresponding algebraic iterated turn $(\epsilon, s_1^{-1}s_2; \varphi_\circ|_{G_\circ})_{G_\circ}$ is well-defined for $(x_1, x_2; \varphi)_F$ and $\circ \in (\star_1, \star_2)$.

II.3 Coordinate-free laminations

We have only defined the stable laminations for an expanding irreducible train track $[\tau]$ representing $[\psi]$. The free splitting $\mathcal{T}$ of $\mathcal{F}$ can be seen as a *coordinate system* and we need a *coordinate-free* definition of stable laminations that applies to all outer automorphisms.

Fix a proper free factor system $\mathcal{Z}$ of $\mathcal{F}$ and consider the set $scv(\mathcal{F}, \mathcal{Z})$ of all free splittings $\mathcal{T}'$ of $\mathcal{F}$ with $\mathcal{F}[\mathcal{T}'] = \mathcal{Z}$, i.e. an element of $\mathcal{F}$ is $\mathcal{T}'$-elliptic if and only if it is conjugate to an element of $\mathcal{Z}$; this set with some natural partial order is the *spine of relative outer space* [CV86]. For any pair of free splittings $\mathcal{T}_1, \mathcal{T}_2 \in scv(\mathcal{F}, \mathcal{Z})$, there are changes of coordinates, equivariant PL-maps $\mathcal{T}_1 \rightleftarrows \mathcal{T}_2$. We saw in the discussion following Claim I.5 that a change of coordinates $f\colon \mathcal{T}_1 \to \mathcal{T}_2$ induces a canonical homeomorphism $f_*\colon \mathbb{R}(\mathcal{T}_1) \to \mathbb{R}(\mathcal{T}_2)$ on the space of lines. Denote the canonical homeomorphism class of $\mathbb{R}(\mathcal{T}_1) \cong \mathbb{R}(\mathcal{T}_2)$ by $\mathbb{R}(\mathcal{F}, \mathcal{Z})$. If $\mathcal{Z}$ is the trivial free factor system, then we denoted the canonical homeomorphism class by $\mathbb{R}(\mathcal{F})$ instead.

Fix an automorphism $\psi\colon \mathcal{F} \to \mathcal{F}$ and a $[\psi]$-invariant proper free factor system $\mathcal{Z}$. Let $\psi_*\colon \mathbb{R}(\mathcal{F}, \mathcal{Z}) \to \mathbb{R}(\mathcal{F}, \mathcal{Z})$ be the *canonical* induced homeomorphism on the space of lines: $f_* \circ g_{1*} = g_{2*} \circ f_*$ for any $\mathcal{T}_1, \mathcal{T}_2 \in scv(\mathcal{F}, \mathcal{Z})$, equivariant PL-map $f\colon \mathcal{T}_1 \to \mathcal{T}_2$, and ψ-equivariant PL-maps $g_i\colon \mathcal{T}_i \to \mathcal{T}_i$ $(i = 1, 2)$. A line $[l] \in \mathbb{R}(\mathcal{F}, \mathcal{Z})$ weakly ψ_*-limits to a lamination $\Lambda \subset \mathbb{R}(\mathcal{F}, \mathcal{Z})$ if the sequence $(\psi_*^n[l])_{n\geq 0}$ weakly limits to Λ.

A coordinate-free definition of stable laminations boils down to characterizing the lines of a stable $\mathcal{T}$-lamination for $[\tau]$ in a way that is independent of coordinates. For the rest of the section, assume there is an equivariant PL-map $(\mathcal{T}, d) \to (\mathcal{Y}, \delta)$ and consider the canonical embedding $\mathbb{R}(\mathcal{Y}, \delta) \subset \mathbb{R}(\mathcal{T})$. Note that lamination $\Lambda \subset \mathbb{R}(\mathcal{Y}, \delta)$ is contained in a canonical lamination $\mathcal{L} \subset \mathbb{R}(\mathcal{T})$: set $\mathcal{L}$ to be the closure of Λ in $\mathbb{R}(\mathcal{T})$.

Claim (cf. [BFH97, Lemma 1.9(2)])**.** *A line is quasiperiodic in* $\mathbb{R}(\mathcal{Y}, \delta)$ *if it is quasiperiodic in* $\mathbb{R}(\mathcal{T})$. (exercise) □

So quasiperiodicity is a well-defined property for a line in $\mathbb{R}(\mathcal{F}, \mathcal{Z})$; moreover, the induced homeomorphism $\psi_*\colon \mathbb{R}(\mathcal{F}, \mathcal{Z}) \to \mathbb{R}(\mathcal{F}, \mathcal{Z})$ preserves quasiperiodicity for any automorphism $\psi\colon \mathcal{F} \to \mathcal{F}$ that has $\mathcal{Z}$ as a $[\psi]$-invariant proper free factor system.

Suppose there is an expanding irreducible train track $\tau\colon \mathcal{T} \to \mathcal{T}$ for ψ with $\mathcal{F}[\mathcal{T}] = \mathcal{Z}$. Recall that the eigenlines of $[\tau^k]$ (for some $k \geq 1$) are constructed by iterating an expanding edge; more precisely, an eigenline $[l]$ of $[\tau^k]$ is the union $\bigcup_{n\geq 1} \tau^{kn}(\mathcal{F} \cdot e)$ for some edge $e \subset l$. The leaf segments $\tau^{kn}(e)$ determine a neighbourhood basis for $[l]$ in the space of lines.

For a line $[l] \in \mathbb{R}(\mathcal{F}, \mathcal{Z})$, a subset $U \subset \mathbb{R}(\mathcal{F}, \mathcal{Z})$ is a ψ_*^k-attracting neighbourhood of $[l]$ if $\psi_*^k(U) \subset U$ and $\{\psi_*^{kn}(U) : n \geq 1\}$ is a neighbourhood basis for $[l]$ in the space of lines. A stable lamination for $[\psi]$ rel. $\mathcal{Z}$ is the closure of a quasiperiodic line in $\mathbb{R}(\mathcal{F}, \mathcal{Z})$ with a ψ_*^k-attracting neighbourhood for some $k \geq 1$. Note that the homeomorphism $\psi_* \colon \mathbb{R}(\mathcal{F}, \mathcal{Z}) \to \mathbb{R}(\mathcal{F}, \mathcal{Z})$ permutes the stable laminations for $[\psi]$ rel. $\mathcal{Z}$ and, by Lemma II.1, each stable $\mathcal{T}$-lamination for $[\tau]$ is identified with some stable lamination for $[\psi]$ rel. $\mathcal{Z}$. Let $\mathcal{L}_{\mathcal{Z}}^+[\psi]$ be the union of stable laminations for $[\psi]$ rel. $\mathcal{Z}$.

Lemma II.5 (cf. [BFH97, Lemma 1.12])**.** *Let $\psi \colon \mathcal{F} \to \mathcal{F}$ be an automorphism, $\tau \colon \mathcal{T} \to \mathcal{T}$ an expanding irreducible train track for ψ, and $\mathcal{Z} \coloneqq \mathcal{F}[\mathcal{T}]$. The stable laminations $\mathcal{L}^+[\tau]$ for $[\tau]$ are identified with the stable laminations $\mathcal{L}_{\mathcal{Z}}^+[\psi]$ for $[\psi]$ rel. $\mathcal{Z}$.*

So $\mathcal{L}_{\mathcal{Z}}^+[\psi]$ is a lamination system whose finitely many components are the stable laminations for $[\psi]$ rel. $\mathcal{Z}$ and these are transitively permuted by $\psi_* \colon \mathbb{R}(\mathcal{F}, \mathcal{Z}) \to \mathbb{R}(\mathcal{F}, \mathcal{Z})$.

Sketch of proof. Suppose a line $[l]$ in $\mathcal{T}$ has a τ_*^k-attracting neighbourhood U for some $k \geq 1$. This forces any $\mathcal{T}$-loxodromic conjugacy class $[x]$ with axis in U to have a translation distance that (eventually) grows under forward $[\psi^k]$-iteration. In particular, the conjugacy class $[x]$ is $\mathcal{Y}_\tau$-loxodromic and the line $[l]$, a weak ψ_*^k-limit of the $\mathcal{T}$-axis for $[x]$, is a leaf in $\mathcal{L}^+[\tau]$ by Proposition II.2. □

The stable laminations $\mathcal{L}_{\mathcal{Z}}^+[\psi]$ are in the subspace $\mathbb{R}(\mathcal{Y}_\tau, d_\infty) \subset \mathbb{R}(\mathcal{F}, \mathcal{Z})$.

II.4 Constructing limit forests (2)

This chapter has thus far focused on automorphims with expanding irreducible train tracks. For the rest of the chapter, we extend our focus to all automorphisms.

Fix an automorphism $\psi \colon \mathcal{F} \to \mathcal{F}$ and set $\mathcal{F}_1 \coloneqq \mathcal{F}$, $\psi_1 \coloneqq \psi$. By Theorem I.1, there is an irreducible train track $\tau_1 \colon \mathcal{T}_1 \to \mathcal{T}_1$ for ψ_1 (use the trivial $\mathcal{Z}$). By ψ_1-equivariance of τ_1, the nontrivial vertex stabilizers of $\mathcal{T}_1$ determine a $[\psi_1]$-invariant proper free factor system $\mathcal{F}_2 \coloneqq \mathcal{F}[\mathcal{T}_1]$. The restriction of ψ_1 to $\mathcal{F}_2$ determines a unique outer class of automorphisms $\psi_2 \colon \mathcal{F}_2 \to \mathcal{F}_2$. We can repeatedly apply Theorem I.1 to ψ_{i+1} $(i \geq 1)$ as long as $\lambda[\tau_i] = 1$ and $\mathcal{F}_{i+1} \coloneqq \mathcal{F}[\mathcal{T}_i]$ contains a noncyclic component. Bass-Serre theory implies this process must stop; we end up with a maximal sequence $(\tau_i)_{i=1}^n$ of irreducible train tracks with $\lambda[\tau_i] = 1$ for $1 \leq i < n$ — such a maximal sequence is called a descending sequence for $[\psi]$ rel. $\mathcal{F}_{n+1}$.

This leads to our *working definition* of growth type: $[\psi]$ is polynomially growing if and only if $\lambda[\tau_n] = 1$ [Mut21b, Proposition III.1]. For any polynomially growing automorphism, define the limit forest to be degenerate.

Suppose $[\psi]$ is exponentially growing and $(\tau_i \colon \mathcal{T}_i \to \mathcal{T}_i)_{i=1}^n$ is a descending sequence of irreducible train tracks for $[\psi]$. Sections II.1–II.2 already cover the case $n = 1$, so we may assume $n > 1$ for the rest of the chapter. Set $\lambda \coloneqq \lambda[\tau_n] > 1$, $\mathcal{T}_n^\circ \coloneqq \mathcal{T}_n$, $\tau_n^\circ \coloneqq \tau_n$, and d_n° the eigenmetric on $\mathcal{T}_n^\circ$ for τ_n°. For $1 \leq i < n$, we inductively form an equivariant *simplicial* *blow-up* $\mathcal{T}_i^\circ$ of $\mathcal{T}_i$ rel. $\mathcal{T}_{i+1}^\circ$: the vertices with nontrivial stabilizers are equivariantly replaced by copies of corresponding components of $\mathcal{T}_{i+1}^\circ$ and arbitrary vertices in $\mathcal{T}_{i+1}^\circ$ are chosen as attaching points for the edges of $\mathcal{T}_i$. Let $\tau_i^\circ\colon \mathcal{T}_i^\circ \to \mathcal{T}_i^\circ$ be the topological representative for ψ_i induced by τ_i and τ_{i+1}°. As τ_i is a simplicial automorphism, we can make τ_i° a λ-Lipschitz map, when extending the metric d_{i+1}° on $\mathcal{T}_{i+1}^\circ$ to a metric d_i° on $\mathcal{T}_i^\circ$, by assigning the same large enough length to the edges of $\mathcal{T}_i$ in the blow-up $\mathcal{T}_i^\circ$. The topological representative $\tau^\circ := \tau_i^\circ$ on $\mathcal{T}^\circ := \mathcal{T}_1^\circ$ is an equivariant blow-up of the descending sequence $(\tau_i)_{i=1}^n$. Set $d^\circ := d_1^\circ$ and identify $(\mathcal{T}_i^\circ, d_i^\circ)$ with the characteristic subforest of $(\mathcal{T}^\circ, d^\circ)$ for $\mathcal{F}_i$. Translates of edges in $\mathcal{T}^\circ$ coming from $\mathcal{T}_i$ form the i^{th} stratum of $\mathcal{T}^\circ$: the n^{th} stratum is exponential while the rest are (relatively) polynomial.

As in Section II.1, the equivariant metric maps $\tau^{\circ m}\colon (\mathcal{T}^\circ, d^\circ) \to (\mathcal{T}^\circ\psi^m, \lambda^{-m}d^\circ)$ converge (as $m \to \infty$) to an equivariant metric surjection $\pi^\circ\colon (\mathcal{T}^\circ, d^\circ) \to (\mathcal{Y}, \delta)$. The map τ° induces a ψ-equivariant λ-homothety $h\colon (\mathcal{Y}, \delta) \to (\mathcal{Y}, \delta)$ and π° semiconjugates τ° to h: $\pi^\circ \circ \tau^\circ = h \circ \pi^\circ$. By restricting to $\mathcal{T}_i^\circ$, we have also constructed the equivariant metric surjection $\pi_i^\circ\colon (\mathcal{T}_i^\circ, d^\circ) \to (\mathcal{Y}_i, \delta)$ and ψ_i-equivariant λ-homothety h_i on $(\mathcal{Y}_i, \delta)$ for $2 \le i \le n$.

The $\mathcal{F}_n$-forest $(\mathcal{Y}_n, \delta)$ is the limit forest for $[\tau_n^\circ]$; so it is a nondegenerate minimal $\mathcal{F}_n$-forest with trivial arc stabilizers. For induction, assume $(\mathcal{Y}_i, \delta)$ is a nondegenerate minimal $\mathcal{F}_i$-forest with trivial arc stabilizers for $2 \le i \le n$. Equivariantly collapse $\mathcal{T}_2^\circ$ in $(\mathcal{T}^\circ, d^\circ)$ to get the $\mathcal{F}$-forest $(\mathcal{T}_1, d^\circ)$. For $m \ge 0$, the metric free splitting $(\mathcal{T}^\circ\psi^m, \lambda^{-m}d^\circ)$ is an equivariant *metric* blow-up of $(\mathcal{T}_1\psi^m, \lambda^{-m}d^\circ)$ rel. $(\mathcal{T}_2^\circ\psi_2^m, \lambda^{-m}d^\circ)$. Since $\tau_1\colon (\mathcal{T}_1, d^\circ) \to (\mathcal{T}_1\psi, d^\circ)$ is an equivariant isometry, the limit $(\mathcal{Y}, \delta)$ is equivariantly isometric to an equivariant metric blow-up of $(\mathcal{T}_1, d^\circ)$ rel. $(\mathcal{Y}_2, \delta)$ whose top stratum (edges coming from $\mathcal{T}_1$) have then been equivariantly collapsed, also known as a *graph of actions (with degenerate skeleton)* — more details are given in the next subsection. Thus $(\mathcal{Y}, \delta)$ is a nondegenerate minimal $\mathcal{F}$-forest with trivial arc stabilizers. See [Mut21b, Theorem IV.1] for a direct construction of $(\mathcal{Y}, \delta)$. This sketches the general case of Proposition I.2. The $\mathcal{F}$-forest $(\mathcal{Y}, \delta)$ is the limit forest for $[\tau_i]_{i=1}^n$.

II.4.1 Decomposing limit forests

We now give a hierarchical decomposition of the limit forest $(\mathcal{Y}, \delta)$ and its space of lines.

Suppose some iterate $[\tau_1^k]$ fixes all $\mathcal{F}$-orbits of vertices and edges in $\mathcal{T}_1$. Fix an edge e in $\mathcal{T}_1$ and one of its endpoints p. Identifying $(\mathcal{T}^\circ\psi^{mk}, \lambda^{-mk}d^\circ)$ with the equivariant metric blow-up of $(\mathcal{T}_1, \lambda^{-mk}d^\circ)$ rel. $(\mathcal{T}_2^\circ\psi_2^{mk}, \lambda^{-mk}d^\circ)$ for $m \ge 0$, we let $p_m \in \mathcal{T}_2^\circ\psi_2^{mk}$ be the attaching point of e to $\mathcal{T}_2^\circ\psi_2^{mk}$ corresponding to the endpoint p. Since $[\tau_1^k]$ fixes $[e]$ and $[p]$, we get $p_m = \psi_2^{-k}(s) \cdots \psi_2^{-mk}(s) \cdot p_0$ for $m \ge 1$. By the first part of the proof for Proposition II.4, the sequence $(p_m)_{m\ge0}$ converges to the unique fixed point $\star$ of $s^{-1} \cdot h_2^k$ in the metric completion $(\overline{\mathcal{Y}}_2, \delta)$. So, in the description of $(\mathcal{Y}, \delta)$ as a graph of actions, the edge e is identified with $\star$. Thus the closure $\widehat{\mathcal{Y}}_2$ of $\mathcal{Y}_2$ in $(\mathcal{Y}, \delta)$ is the union of $\mathcal{Y}_2$ with the $\mathcal{F}_2$-orbits of attaching points $\star$ as the pair (e, p) ranges over the $\mathcal{F}$-orbit representatives e of edges and their endpoints p. For the same reasons, we inductively get a similar description of the closure $\widehat{\mathcal{Y}}_{i+1}$ of $\mathcal{Y}_{i+1}$ in $(\mathcal{Y}_i, \delta)$ for $2 \le i < n$.

For $1 < i \le n$, any two translates of $\mathcal{T}_i^\circ \subset \mathcal{T}^\circ$ by elements of $\mathcal{F}$ either coincide or are disjoint by construction. This induces a canonical closed embedding of $\mathbb{R}(\mathcal{F}_i, \mathcal{Z})$ into $\mathbb{R}(\mathcal{F}, \mathcal{Z})$ (exercise). Similarly, any two intersecting translates of $\mathcal{Y}_i \subset \mathcal{Y}$ by elements of $\mathcal{F}$ either coincide or have degenerate intersection. This also induces a canonical closed embedding $\mathbb{R}(\mathcal{Y}_i, \delta) \subset \mathbb{R}(\mathcal{Y}, \delta)$. Finally, the constructed equivariant metric map π° induces a canonical embedding of the topological pair $(\mathbb{R}(\mathcal{Y}, \delta), \mathbb{R}(\mathcal{Y}_i, \delta))$ into $(\mathbb{R}(\mathcal{F}, \mathcal{Z}), \mathbb{R}(\mathcal{F}_i, \mathcal{Z}))$.

Remark. Constructing $(\mathcal{Y}, \delta)$ directly by iterating τ° allows us to lift metric properties of $(\mathcal{Y}, \delta)$ to dynamical properties of τ° through the semiconjugacy $\pi^\circ \circ \tau^\circ = h \circ \pi^\circ$; this viewpoint is used in the next section. On the other hand, constructing $(\mathcal{Y}, \delta)$ directly as we did in [Mut21b, Theorem IV.1] (and hinted at in this subsection) gives us a nice structural description of intervals in the limit forest. This is explained in the next subsection and will be a key component of Chapter III!

II.4.2 Intervals in limit forests

Here is an inductive description of intervals in the limit forest $(\mathcal{Y}, \delta)$ in terms of the limit forest for $[\tau_n^\circ]$. For $1 \le i \le n$, the characteristic subforest $(\mathcal{Y}_i, \delta)$ of $(\mathcal{Y}, \delta)$ for $\mathcal{F}_i$ is the limit forest for $[\tau_j]_{j=i}^n$. For $1 < i \le n$, let $\widehat{\mathcal{Y}}_i$ be the closure of $\mathcal{Y}_i$ in $(\mathcal{Y}_{i-1}, \delta)$.

It follows from the blow-up (and collapse) description of $\mathcal{Y}_{i-1}$ that its closed intervals are finite concatenations of closed intervals in translates of $\widehat{\mathcal{Y}}_i$. As shown in the previous subsection, the $\mathcal{F}_i$-orbits $[p]$ of points in $\widehat{\mathcal{Y}}_i \setminus \mathcal{Y}_i$ are fixed by the extension of $[h_i^k]$ to $\mathcal{F}_i \backslash \widehat{\mathcal{Y}}_i$ for some $k \ge 1$. As $p \notin \mathcal{Y}_i$, it has exactly one direction d_p in $\widehat{\mathcal{Y}}_i$. This direction's $\mathcal{F}_i$-orbit $[d_p]$ is also fixed (setwise) by the expanding homothety h_i^k and d_p determines a singular eigenray $\rho_p \subset \widehat{\mathcal{Y}}_i$ of $[h_i^k]$ based at p. For any point $q \in \mathcal{Y}_i$, the closed interval $[p, q] \subset \widehat{\mathcal{Y}}_i$ is a concatenation of an initial segment of the singular eigenray ρ_p and a closed interval in $\mathcal{Y}_i$; therefore, closed intervals in $\mathcal{Y}_{i-1}$ are finite concatenations of translates of closed intervals in $\mathcal{Y}_i$ and initial segments of singular eigenrays of $[h_i^k]$ for some $k \ge 1$.

Let $\mathcal{L}^+_{\mathcal{Z}}[\psi_n] = \mathcal{L}^+[\tau_n]$ be the k-component stable laminations for $[\tau_n^\circ] = [\tau_n]$ and $\oplus_{j=1}^k \delta_j$ the factored $\mathcal{F}_n$-invariant convex metric on $\mathcal{Y}_n$ indexed by components $\Lambda_j^+ \subset \mathcal{L}^+_{\mathcal{Z}}[\psi_n]$. By the inductive description of intervals in $\mathcal{Y}$ and the fact h_n^k is a λ^k-homothety with respect to each factor δ_j, we get: δ_j equivariantly extends to $\mathcal{Y}$; $\delta = \oplus_{j=1}^k \delta_j$ is a factored $\mathcal{F}$-invariant convex metric on $\mathcal{Y}$; and h^k is a λ^k-homothety with respect to each factor δ_j.

The lamination $\mathcal{L}^+_{\mathcal{Z}}[\psi_n] \subset \mathbb{R}(\mathcal{Y}_n, \delta)$ can be seen as a $(\mathcal{Y}, \delta)$-lamination since $\mathbb{R}(\mathcal{Y}_n, \delta)$ is a closed subspace of $\mathbb{R}(\mathcal{Y}, \delta)$. Note that closed edges of $\mathcal{T}_n$ are leaf segments (of $\mathcal{L}^+_{\mathcal{Z}}[\psi_n]$); thus any closed interval in $\mathcal{T}_n^\circ$ is a finite concatenation of leaf segments. As the equivariant PL-map $\pi_n^\circ \colon (\mathcal{T}_n^\circ, d^\circ) \to (\mathcal{Y}_n, \delta)$ is surjective and isometric on leaf segments, we get:

Lemma II.6. *Let $\tau_n \colon \mathcal{T}_n \to \mathcal{T}_n$ be an expanding irreducible train track and $(\mathcal{Y}_n, \delta)$ its limit forest. Any closed interval in $\mathcal{Y}_n$ is a finite concatenation of leaf segments of $\mathcal{L}^+[\tau_n]$.* □

This lemma no longer holds when $n \geq 2$ and we consider closed intervals in $\widehat{\mathcal{Y}}_n$. To account for this, let n^{th} level leaf blocks in $\mathcal{Y}$ be leaf segments. By the lemma, any interval of $\mathcal{Y}_n$ is a finite concatenation of n^{th} level leaf blocks.

Inductively define the $(i-1)^{st}$ level leaf blocks in $\mathcal{Y}$ $(1 < i \leq n)$ to be the i^{th} level leaf blocks or (translates of) closed intervals in singular eigenrays $\rho \subset \widehat{\mathcal{Y}}_i$ of $[h_i]$-iterates. By the earlier description of intervals and induction hypothesis, any interval of $\mathcal{Y}_{i-1}$ is a finite concatenation of $(i-1)^{st}$ level leaf blocks. The 1^{st} level leaf blocks are simply leaf blocks. Altogether, we have proven a generalization of Lemma II.6 in terms of leaf blocks:

Lemma II.7. *Let $(\tau_i \colon \mathcal{T}_i \to \mathcal{T}_i)_{i=1}^n$ be a descending sequence of irreducible train tracks for an automorphism $\psi \colon \mathcal{F} \to \mathcal{F}$ and $(\mathcal{Y}, \delta)$ the corresponding limit forest. Any closed interval in $\mathcal{Y}$ is a finite concatenation of leaf blocks of $\mathcal{L}^+_{\mathcal{Z}}[\psi_n]$.* □

II.5 Stable laminations (2)

Fix an exponentially growing automorphism $\psi \colon \mathcal{F} \to \mathcal{F}$ with a descending sequence $(\tau_i \colon \mathcal{T}_i \to \mathcal{T}_i)_{i=1}^n$ of irreducible train tracks. Let $(\mathcal{Y}, \delta)$ be the limit forest for $[\tau_i]_{i=1}^n$, $\mathcal{T}^\circ$ an equivariant blow-up of free splittings $(\mathcal{T}_i)_{i=1}^n$, and $\mathcal{Z} \coloneqq \mathcal{F}[\mathcal{T}^\circ]$. The free splitting $\mathcal{T}_n$ of $\mathcal{F}_n \coloneqq \mathcal{F}[\mathcal{T}_{n-1}]$ is identified with the characteristic subforest of $(\mathcal{T}^\circ, d^\circ)$ for $\mathcal{F}_n$.

Claim II.8. *The stable laminations $\mathcal{L}^+_{\mathcal{Z}}[\psi_n]$ for $[\psi_n]$ in $\mathbb{R}(\mathcal{F}_n, \mathcal{Z})$ are identified with the stable laminations $\mathcal{L}^+_{\mathcal{Z}}[\psi]$ for $[\psi]$ in $\mathbb{R}(\mathcal{F}, \mathcal{Z})$.*

Note that $\mathcal{L}^+_{\mathcal{Z}}[\psi] = \mathcal{L}^+_{\mathcal{Z}}[\psi_n]$ is in the subspace $\mathbb{R}(\mathcal{Y}_n, \delta) \subset \mathbb{R}(\mathcal{Y}, \delta) \subset \mathbb{R}(\mathcal{F}, \mathcal{Z})$.

Sketch of proof. Since $\lambda[\tau_i] = 1$ for $i < n$, no birecurrent line in $\mathbb{R}(\mathcal{F}, \mathcal{F}_n)$ has a ψ^k_*-attracting neighbourhood for any $k \geq 1$. Thus any stable lamination for $[\psi]$ in $\mathbb{R}(\mathcal{F}, \mathcal{Z})$ is contained in $\mathbb{R}(\mathcal{F}_n, \mathcal{Z})$ and corresponds to a stable lamination for $[\psi_n]$. □

We generalize Proposition II.4 by characterizing limits of iterated turns over $\mathcal{T}^\circ$:

Theorem II.9. *Let $\psi \colon \mathcal{F} \to \mathcal{F}$ be an exponentially growing automorphism with a descending sequence $(\tau_i \colon \mathcal{T}_i \to \mathcal{T}_i)_{i=1}^n$ of irreducible train tracks rel. $\mathcal{Z}$, $(\mathcal{Y}, \delta)$ the limit forest for $[\tau_i]_{i=1}^n$, and $\mathcal{T}^\circ$ an equivariant blow-up of free splittings $(\mathcal{T}_i)_{i=1}^n$. Choose a nondegenerate component $T^\circ \subset \mathcal{T}^\circ$, corresponding components $F \subset \mathcal{F}$, $Y \subset \mathcal{Y}$, and a positive iterate ψ^k that preserves F. Let $\tilde{h} \colon (Y, \delta) \to (Y, \delta)$ be the φ-equivariant λ-homothety, where φ is in the outer class $[\psi^k|_F]$ and $\lambda \coloneqq (\lambda[\tau_n])^k$. Finally, for $\iota = 1, 2$, pick $p_\iota \in T^\circ$ and $x_\iota \in F$.*

The point $p_{\iota,m} \coloneqq \varphi^{-1}(x_\iota) \cdots \varphi^{-m}(x_\iota) \cdot p_\iota$ in $(T^\circ \varphi^m, \lambda^{-m} d^\circ)$ converges to $\star_\iota$ in $(\overline{Y}, \delta)$ as $m \to \infty$, where $\star_\iota$ is the unique fixed point of $x_\iota^{-1} \cdot \tilde{h}$ in the metric completion $(\overline{Y}, \delta)$.

If $x_1^{-1} x_2$ fixes $\star_1$, then $\star_1 = \star_2$ and the term $[p_{1,m}, p_{2,m}]$ $(m \geq 0)$ of the iterated turn $(p_1, p_2 : x_1, x_2; \varphi)_{T^\circ}$ has combinatorial length *$\leq \alpha(m)$ for some (degree n) polynomial α. Otherwise, $\star_1 \neq \star_2$ and the iterated turn weakly limits to a component of $\mathcal{L}^+_{\mathcal{Z}}[\psi]$.*

An interval $[p,q] \subset T^\circ$ has combinatorial length $N+1$ if N is the number of vertices in the open interval (p,q). An iterated turn $[p_{1,m}, p_{2,m}]_{m\geq 0}$ weakly limits to a component $\Lambda^+ \subset \mathcal{L}^+_{\mathcal{Z}}[\psi]$ if the term $[p_{1,m}, p_{2,m}]$ contains a leaf segment of Λ^+ with arbitrarily large combinatorial length as $m \to \infty$.

Sketch of proof. Let $\tilde{\tau}^\circ \colon (T^\circ, d^\circ) \to (T^\circ, d^\circ)$ be the φ-equivariant λ-Lipschitz topological representative induced by the irreducible train tracks $(\tau_i)_{i=1}^n$ and $\pi^\circ \colon (T^\circ, d^\circ) \to (Y, \delta)$ the equivariant metric map constructed using $\tilde{\tau}^\circ$-iteration. Even though π° may fail to be a PL-map, it still has a cancellation constant $C[\pi^\circ] \geq 0$ as a limit of equivariant metric maps with uniformly bounded cancellation constants. The proof of the first part is the same as in Proposition II.4 using π°, $\tilde{\tau}^\circ$, and the φ-equivariant λ-homothety $\tilde{h}$.

The interval $[p_{1,m}, p_{2,m}] \subset T^\circ$, a term in the sequence $(p_1, p_2 : x_1, x_2; \varphi)_{T^\circ}$, is covered by certain $2m+1$ intervals as in the proof of Proposition II.4. Since $\tilde{\tau}^\circ$ is induced by a descending sequence $(\tau_i)_{i=1}^n$ of irreducible train tracks, the intervals $[\tilde{\tau}^{\circ(l-1)}(x_1 \cdot p_1), \tilde{\tau}^{\circ l}(p_1)]$, $[\tau^{\circ l}(p_1), \tilde{\tau}^{\circ l}(p_2)]$, and $[\tilde{\tau}^{\circ l}(p_2), \tilde{\tau}^{\circ(l-1)}(x_2 \cdot p_2)]$ are covered by $\alpha(l)$ polynomial strata edges and leaf segments (of $\mathcal{L}^+_{\mathcal{Z}}[\psi]$) for some degree $(n-1)$ polynomial α. So the interval $[p_{1,m}, p_{2,m}]$ is covered by $\alpha(m) + \sum_{l=1}^m 2\alpha(l)$ polynomial strata edges and leaf segments (of $\mathcal{L}^+_{\mathcal{Z}}[\psi]$). Note that $\alpha(m) + \sum_{l=1}^m 2\alpha(l) \leq \beta(m)$ for some degree n polynomial β.

Assume $\star_1 = \star_2$, where $\star_\iota$ is the unique fixed point of $\tilde{h}_\iota \coloneqq x_\iota^{-1} \cdot \tilde{h}$ in metric completion $(\overline{Y}, \delta)$ for $\iota = 1, 2$. The proof given in Proposition II.4 implies there is a uniform bound on the d°-length, and hence combinatorial length, of leaf segments in $[p_{1,m}, p_{2,m}]$. Consequently, the combinatorial length of $[p_{1,m}, p_{2,m}]$ is $\leq \beta(m)B$ for some constant $B \geq 1$.

Assume $\star_1 \neq \star_2$. Set $L \coloneqq \frac{1}{2}\delta(\star_1, \star_2) > 0$; then $\delta(\tilde{h}_1^{-m}(\pi^\circ(p_1)), \tilde{h}_2^{-m}(\pi^\circ(p_2))) > L$ and $d^\circ(p_{1,m}, p_{2,m}) > \lambda^m L$ for $m \gg 1$. The contribution of polynomial strata to the d°-length of $[p_{1,m}, p_{2,m}]$ is at most $\beta(m)B'$ for some constant $B' \geq 1$; the exponential stratum edges intersecting the interval are covered by $\beta(m)$ leaf segments. By the pigeonhole principle, the interval $[p_{1,m}, p_{2,m}]$, a term in the iterated turn $(p_1, p_2 : x_1, x_2; \varphi)_{T^\circ}$, has a leaf segment of d°-length $\geq \frac{\lambda^m L - \beta(m)B'}{\beta(m)} \gg 1$. Quasiperiodicity implies the iterated turn weakly limits to a component of $\mathcal{L}^+_{\mathcal{Z}}[\psi]$. $\square$

Remark. The argument given in Subsection II.2.4 applies in this general context: it describes how an iterated turn over $\mathcal{F}$ determines (nested) iterated turns over $\mathcal{G}[\mathcal{Y}]$.

As in Proposition II.2, we can characterize the elements in $\mathcal{F}$ that are $\mathcal{Y}$-loxodromic:

Theorem II.10. *Let $\psi \colon \mathcal{F} \to \mathcal{F}$ be an exponentially growing automorphism with a descending sequence $(\tau_i \colon \mathcal{T}_i \to \mathcal{T}_i)_{i=1}^n$ of irreducible train tracks rel. $\mathcal{Z}$, $(\mathcal{Y}, \delta)$ the limit forest for $[\tau_i]_{i=1}^n$, and $\mathcal{T}^\circ$ an equivariant blow-up of free splittings $(\mathcal{T}_i)_{i=1}^n$.*

If $x \in \mathcal{F}$ is a $\mathcal{T}^\circ$-loxodromic element, then the following statements are equivalent:

1. *the element x is $\mathcal{Y}$-loxodromic;*

2. *the element x $[\psi]$-grows exponentially rel. $\mathcal{Z}$ with rate $\lambda[\tau_n]$; and*

3. *the axis for the conjugacy class $[x]$ in $\mathbb{R}(\mathcal{F},\mathcal{Z})$ weakly ψ_*-limits to $\mathcal{L}^+_{\mathcal{Z}}[\psi]$.*

The restriction of ψ to the $[\psi]$-invariant subgroup system $\mathcal{G}[\mathcal{Y}]$ of $\mathcal{Y}$-point stabilizers is polynomially growing rel. $\mathcal{Z}$ with degree $< n$.

Sketch of proof. Set $\lambda := \lambda[\tau_n]$, $\mathcal{F}_1 := \mathcal{F}$, and $\mathcal{F}_{i+1} := \mathcal{F}[\mathcal{T}_i]$ for $1 \le i < n$. Under the canonical embedding $\mathbb{R}(\mathcal{F}_i,\mathcal{Z}) \subset \mathbb{R}(\mathcal{F},\mathcal{Z})$, we identify the stable laminations $\mathcal{L}^+_{\mathcal{Z}}[\psi]$ and $\mathcal{L}^+_{\mathcal{Z}}[\psi_i]$. Let $\mathcal{T}^\circ$ be an equivariant blow-up of free splittings $(\mathcal{T}_i)_{i=1}^n$ and $\mathcal{T}_i^\circ \subset \mathcal{T}^\circ$ the characteristic convex subsets for $\mathcal{F}_i$. Suppose $x \in \mathcal{F}_1$ is a $\mathcal{T}^\circ$-loxodromic element. The equivalence between Conditions 1–3 is given by Proposition II.2 if x is conjugate to an element of $\mathcal{F}_n$. Assume $n \ge 2$ and, up to conjugacy, $x \in \mathcal{F}_i$ is $\mathcal{T}_i$-loxodromic for some $i < n$.

Recall that $\tau^\circ \colon (\mathcal{T}^\circ, d^\circ) \to (\mathcal{T}^\circ, d^\circ)$ is a ψ-equivariant λ-Lipschitz topological representative induced by the irreducible train tracks $(\tau_i)_{i=1}^n$ and $\pi^\circ \colon (\mathcal{T}^\circ, d^\circ) \to (\mathcal{Y},\delta)$ the constructed equivariant metric map. In particular, $\limsup_{m\to\infty} \frac{1}{m} \log \|\psi^m(x)\|_{d^\circ} \le \log\lambda$.

Suppose $[\tau_i^k]$ (for some $k \ge 1$) fixes all $\mathcal{F}_i$-orbits of vertices and edges in $\mathcal{T}_i$. Let $l^\circ \subset \mathcal{T}_i^\circ$ be the axis for $x \in \mathcal{F}_i$. The axis l° projects to the axis l of x in $\mathcal{T}_i$; write l as a biinfinite concatenation of edges $\cdots e_{-1} \cdot e_0 \cdot e_1 \cdots$ and identify $e_j \subset \mathcal{T}_i$ with its lift to $\mathcal{T}_i^\circ$. For $m \ge 0$ and any integer j, let $w_{j,m}$ be the closed interval in $\mathcal{T}_i^\circ$ between (lifts of) $\tau_i^m(e_j)$ and $\tau_i^m(e_{j+1})$; in fact, $w_{j,m}$ is in a component of $\mathcal{F}_i \cdot \mathcal{T}_{i+1}^\circ \subset \mathcal{T}_i^\circ$. Since $[\tau_i^k]$ fixes the $\mathcal{F}_i$-orbits $[e], [e']$ and the vertex of $\mathcal{T}_i$ between them, the sequence $(w_{j,mk+r})_{m\ge 0}$, up to translation, is an iterated turn over $\mathcal{T}_{i+1}^\circ$ rel. ψ_{i+1}^k for $0 \le r < k$; by Theorem II.9, the iterated turn limits to an interval $w_{j,r}^*$ in a translate of a component of $\widehat{\mathcal{Y}}_{i+1} \subset \mathcal{Y}_i$.

The intervals $w_{j,m}, w_{j+1,m}$ are always in distinct components of $\mathcal{F}_i \cdot \mathcal{T}_{i+1}^\circ$; therefore, the limit intervals $w_{j,r}^*, w_{j+1,r}^*$ have degenerate intersection. By the equivariance of the limits, the union $l_* := \bigcup_j w_{j,0}^*$ is an x-invariant arc. If some limit interval $w_{j,0}^*$ is not degenerate, then x is $\mathcal{Y}_i$-loxodromic and l_* is its $\mathcal{Y}_i$-axis; otherwise, l_* is degenerate and x is $\mathcal{Y}_i$-elliptic.

<u>Case 1:</u> x is $\mathcal{Y}_i$-loxodromic, i.e. some limit interval $w_{j,0}^*$ is not degenerate. By Theorem II.9, the iterated turn $(w_{j,mk})_{m\ge 0}$ over $\mathcal{T}_{i+1}^\circ$ rel. ψ_{i+1}^k weakly limits to a component of $\mathcal{L}^+_{\mathcal{Z}}[\psi_{i+1}]$. So $[l^\circ] \in \mathbb{R}(\mathcal{F},\mathcal{Z})$ weakly ψ_*^k-limits to a component of $\mathcal{L}^+_{\mathcal{Z}}[\psi]$. Finally, $[l^\circ]$ weakly ψ_*-limits to $\mathcal{L}^+_{\mathcal{Z}}[\psi]$ since ψ_* acts transitively on the components of $\mathcal{L}^+_{\mathcal{Z}}[\psi]$. As π° is an equivariant metric map, $\|\cdot\|_\delta \le \|\cdot\|_{d^\circ}$ and $\log\lambda \le \liminf_{m\to\infty} \frac{1}{m}\log\|\psi^m(x)\|_{d^\circ}$.

<u>Case 2:</u> x is $\mathcal{Y}_i$-elliptic, i.e. each limit interval $w_{j,0}^*$ is degenerate. By Theorem II.9, the interval $w_{j,mk}$ has d°-length is bounded above by some degree $(n-i)$ polynomial (in m). Thus $\|\psi^{mk}(x)\|_{d^\circ}$ is bounded above by a degree $(n-i)$ polynomial. By ψ-equivariance of the homothety h_i, the elements $\psi(x), \ldots, \psi^{k-1}(x)$ are $\mathcal{Y}_i$-elliptic as well. The same argument implies $\|\psi^m(x)\|_{d^\circ}$ is bounded above by a degree $(n-i)$ polynomial. □

We conclude the chapter by stating the extension of Lemma V.3 to all limit forests:

Lemma V.6. *Let $\psi\colon \mathcal{F} \to \mathcal{F}$ be an exponentially growing automorphism with a descending sequence $(\tau_i\colon \mathcal{T}_i \to \mathcal{T}_i)_{i=1}^n$ of irreducible train tracks rel. $\mathcal{Z}$, $(\mathcal{Y}, \delta)$ the limit forest for $[\tau_i]_{i=1}^n$, $\lambda := \lambda[\tau_n]$, and $(\mathcal{Y}', \delta')$ a minimal $\mathcal{F}$-forest with trivial arc stabilizers.*

If $\mathcal{Z}$ is $\mathcal{Y}'$-elliptic and the k-component lamination $\mathcal{L}^+_{\mathcal{Z}}[\psi]$ is in $\mathbb{R}(\mathcal{Y}', \delta') \subset \mathbb{R}(\mathcal{F}, \mathcal{Z})$, then the limit of $(\mathcal{Y}'\psi^{mk}, \lambda^{-mk}\delta')_{m\geq 0}$ is $(\mathcal{Y}, \oplus_{j=1}^k c_j\, \delta_j)$, where $\delta = \oplus_{j=1}^k \delta_j$ and $c_j > 0$.

Again, we postpone the proof to Section V.2. If $(\tau_i')_{i=1}^{n'}$ is another descending sequence for $[\psi]$ rel. $\mathcal{Z}$, then its limit forest $(\mathcal{Y}', \delta')$ is equivariantly homothetic to $(\mathcal{Y}, \delta)$; therefore, $(\mathcal{Y}, \delta)$ is *the* limit forest for $[\psi]$ rel. $\mathcal{Z}$ (up to rescaling of δ). A minimal $\mathcal{F}$-forest $(\mathcal{Y}', \delta')$ with trivial arc stabilizers is an expanding forest for $[\psi]$ rel. $\mathcal{Z}$ if:

1. the $\mathcal{F}$-forest $(\mathcal{Y}', \delta')$ is equivariantly isometric to $(\mathcal{Y}'\psi, s^{-1}\delta')$ for some $s > 0$; and
2. the free factor system $\mathcal{Z}$ is $\mathcal{Y}'$-elliptic.

Corollary II.11. *Let $\psi\colon \mathcal{F} \to \mathcal{F}$ be an exponentially growing automorphism and $(\mathcal{Y}, \delta)$ the limit forest for $[\psi]$ rel. some proper free factor $\mathcal{Z}$. Any expanding forests for $[\psi]$ rel. $\mathcal{Z}$ is uniquely equivariantly homothetic to $(\mathcal{Y}, \delta)$.*

Sketch of proof. Let $(\mathcal{Y}', \delta')$ be an expanding forest for $[\psi]$ rel. $\mathcal{Z}$ and $x \in \mathcal{F}$ a $\mathcal{Y}'$-loxodromic element. The proof is essentially the proof of Corollary II.3 with two main changes. First, choose $m \gg 1$ so that $\|\psi^m(x)\|_{\delta'} > \alpha(m)(2C[f] + B')$ for some polynomial α and constant $B' \geq 1$ determined by x; therefore, a fundamental domain of $\psi^m(x)$ acting on its axis has a leaf segment $[q, r]$ with $\delta'(f(q), f(r)) > 2C[f]$ by the pigeonhole principle. For the second change, conclude the proof by invoking Lemma V.6 instead of Lemma V.3. $\square$

III Topmost limit forests

The limit forest produced by our proof of Proposition I.2 is universal for an outer automorphism and some choice of an invariant proper free factor system $\mathcal{Z}$. The limit forest is degenerate and hence canonical for a polynomially growing automorphism. For the rest of the paper, we consider exponentially growing automorphisms. Our next goal is to construct a *topmost* limit forest that is universal for an outer automorphism.

III.1 Assembling limit hierarchies

This section first summarizes the main result of the paper's prequel [Mut21b]. The general strategy follows closely the construction of limit forests sketched in Section II.4.

Fix an exponentially growing automorphism $\psi\colon \mathcal{F} \to \mathcal{F}$ and set $\mathcal{G}_1 := \mathcal{F}$, $\psi_1 := \psi$. By our proof of Proposition I.2, there is a nondegenerate limit forest $(\mathcal{Y}_1, \delta_1)$ for $[\psi_1]$ rel. $\mathcal{Z}_1$ (some proper free factor system of $\mathcal{G}_1$) and a unique ψ_1-equivariant expanding λ_1-homothety $h_1\colon (\mathcal{Y}_1, \delta_1) \to (\mathcal{Y}_1, \delta_1)$. Thus $\mathcal{Y}_1$-loxodromic elements in $\mathcal{F}$ $[\psi]$-grow exponentially with rate λ_1. By Gaboriau–Levitt index theory and ψ_1-equivariance of τ_1, the nontrivial point stabilizers of $\mathcal{Y}_1$ determine a $[\psi_1]$-invariant malnormal subgroup system $\mathcal{G}_2 := \mathcal{G}[\mathcal{Y}_1]$ with strictly lower complexity than $\mathcal{G}_1$. The restriction of ψ_1 to $\mathcal{G}_2$ determines a unique outer class of automorphisms $\psi_2\colon \mathcal{G}_2 \to \mathcal{G}_2$.

We can repeated apply Proposition I.2 to ψ_{i+1} $(i \geq 1)$ as long as ψ_{i+1} is exponentially growing. This inductive invokation of Proposition I.2 eventually stops since the complexity of $\mathcal{G}_i$ is strictly decreasing (in i) positive integer. In the end, we have a maximal sequence $(\mathcal{Y}_i, \delta_i)_{i=1}^{n}$ of nondegenerate limit forests for $[\psi_i]$ rel. $\mathcal{Z}_i$ each with a unique ψ_i-equivariant expanding λ_i-homothety h_i on $(\mathcal{Y}_i, d_i)$ — such a maximal sequence of limit forests is a descending sequence for $[\psi]$. By construction, an element $x \in \mathcal{F}$ has a conjugate in $\mathcal{G}_{n+1}$ if and only if x $[\psi]$-grows polynomially!

In Section II.4, the blow-ups of free splittings $(\mathcal{T}_i)_{i=1}^{n}$ were arbitary and done inductively upwards (i.e. started with $i = n$). We then used a limiting argument to produce the final limit forest $(\mathcal{Y}, \delta)$. For this section, the blow-ups of limit forests $(\mathcal{Y}_i, \delta_i)_{i=1}^{n}$ will not be arbitrary but will make use of the expanding homotheties $(h_i)_{i=1}^{n}$; moreover, it will be done inductively downwards (i.e. starts with $i = 1$) to produce an $\mathcal{F}$-pseudoforest $(\mathcal{T}, (\delta_i)_{i=1}^{n})$.

Set $(\mathcal{X}^{(1)}, \delta_1) := (\mathcal{Y}_1, \delta_1)$ and $g^{(1)} := h_1$. For $1 < i \leq n$, we inductively construct the *equivariant pseudoforest blow-up* $(\mathcal{X}^{(i)}, (\delta_j)_{j=1}^{i})$ of the $\mathcal{F}$-pseudoforest $(\mathcal{X}^{(i-1)}, (\delta_j)_{j=1}^{i-1})$ rel. the $\mathcal{G}_i$-forest $(\mathcal{Y}_i, \delta_i)$ and expanding homotheties $g^{(i-1)}$ and h_i. Here is a sketch:

Let $(\overline{\mathcal{Y}}_i, \delta_i)$ be the metric completion and $\bar{h}_i$ the extension to the metric completion. For $1 \leq j < i$, assume that $(\mathcal{Y}_i, \delta_i)$ is equivariantly isometric to the associated $\mathcal{G}_j$-forest for the $\mathcal{G}_j$-invariant convex pseudometric d_i restricted to $\mathcal{X}^{(i-1)}(\mathcal{G}_j)$, the characteristic convex subsets of $\mathcal{X}^{(i-1)}$ for $\mathcal{G}_j$. Since the hierarchy $(\delta_j)_{j=1}^{i-1}$ has full support, $(\mathcal{X}^{(i-1)}(\mathcal{G}_{i-1}), \delta_{i-1})$ is equivariantly isometric to $(\mathcal{Y}_i, \delta_i)$ and the nontrivial point stabilizers of $\mathcal{X}^{(i-1)}$ are conjugates in $\mathcal{F}$ of $\mathcal{G}_i$-components. The points of $\mathcal{X}^{(i-1)}$ with nontrivial stabilizers are replaced by corresponding copies of $\overline{\mathcal{Y}}_i$-components; this produces a unique set system $\widehat{\mathcal{X}}^{(i)}$ with an $\mathcal{F}$-action that is the *equivariant set blow-up* of $\mathcal{X}^{(i-1)}$ rel. $\overline{\mathcal{Y}}_i$: it comes with an equivariant injection $\iota_i \colon \overline{\mathcal{Y}}_i \to \widehat{\mathcal{X}}^{(i)}$ and an equivariant surjection $\kappa_i \colon \widehat{\mathcal{X}}^{(i)} \to \mathcal{X}^{(i-1)}$ that is a bijection on the complement $\widehat{\mathcal{X}}^{(i)} \setminus \mathcal{F} \cdot \iota_i(\overline{\mathcal{Y}}_i)$. Consequently, there is a unique ψ-equivariant induced permutation $g^{(i)} \colon \widehat{\mathcal{X}}^{(i)} \to \widehat{\mathcal{X}}^{(i)}$ induced by $g^{(i-1)}$ and $\bar{h}_i$ — κ_i semiconjugates $\hat{g}^{(i)}$ to $g^{(i-1)}$ while ι_i conjugates $\bar{h}_i$ to the restriction $g^{(i)}\big|_{\iota_i(\overline{\mathcal{Y}}_i)}$.

There are plenty of equivariant interval functions $[\cdot,\cdot]^{(i)}$ on $\widehat{\mathcal{X}}^{(i)}$ *compatible with* $\mathcal{X}^{(i-1)}$ and $\mathcal{Y}_i$ — compatibility means the injection ι_i and surjection κ_i map intervals to intervals. Some compatible $\mathcal{F}$-pretrees $(\widehat{\mathcal{X}}^{(i)}, [\cdot,\cdot]^{(i)})$ are real [Mut21b, Proposition IV.3] and they naturally inherit an $\mathcal{F}$-invariant hierarchy $(\hat{\delta}_j)_{j=1}^{i}$ with full support: $(\hat{\delta}_j)_{j=1}^{i-1}$ is the pullback $\kappa_i^*(\delta_j)_{j=1}^{i-1}$ and $\hat{\delta}_i$ is the pushforward $\iota_{i*}\delta_i$ extended equivariantly to the orbit $\mathcal{F} \cdot \iota_i(\overline{\mathcal{Y}}_i)$; moreover, for $1 \leq j \leq i$, $(\mathcal{Y}_j, \delta_j)$ is equivariantly isometric to the associated $\mathcal{G}_j$-forest for the $\mathcal{G}_j$-invariant convex pseudometric $\hat{\delta}_j$ restricted to $\widehat{\mathcal{X}}^{(i)}(\mathcal{G}_j)$.

Claim ([Mut21b, Theorem IV.4])**.** *Since $\bar{h}_i$ is expanding, the permutation $g^{(i)}$ is a pretree-automorphism for a unique real compatible $\mathcal{F}$-pretree $(\widehat{\mathcal{X}}^{(i)}, [\cdot,\cdot]_g^{(i)})$.*

Remark. This is the main technical result of [Mut21b]. Its proof uses Gaboriau–Levitt's index inequality and the contraction mapping theorem.

We now fix the interval function $[\cdot,\cdot]_g^{(i)}$ but omit it for brevity. By construction, the $\mathcal{F}$-pseudoforest $(\widehat{\mathcal{X}}^{(i)}, (\hat{\delta}_j)_{j=1}^{i})$ has trivial arc stabilizers and $g^{(i)}$ is an expanding homothety with respect to $(\hat{\delta}_j)_{j=1}^{i}$. Finally, let $\mathcal{X}^{(i)} \subset \widehat{\mathcal{X}}^{(i)}$ be the characteristic convex subsets for $\mathcal{F}$ and $(\delta_j)_{j=1}^{i}$ the restriction of the hierarchy $(\hat{\delta}_j)_{j=1}^{i}$ to $\mathcal{X}^{(i)}$, then replace the maps ι_i, κ_i, and $g^{(i)}$ with their restrictions to $\mathcal{X}^{(i)}$; so $(\mathcal{X}^{(i)}, (\delta_j)_{j=1}^{i})$ is a minimal $\mathcal{F}$-pseudoforest.

At the n^{th} iteration, we have a minimal $\mathcal{F}$-pseudoforest $(\mathcal{T}, (\delta_i)_{i=1}^{n}) \coloneqq (\mathcal{X}^{(n)}, (\delta_i)_{i=1}^{n})$ with trivial arc stabilizers, unique for the descending sequence $(\mathcal{Y}_i, \delta_i)_{i=1}^{n}$; the ψ-equivariant pretree-automorphism $f \coloneqq g^{(n)}$ on $(\mathcal{T}, (\delta_i)_{i=1}^{n})$ is a $(\lambda_i)_{i=1}^{n}$-homothety, where $\lambda_i > 1$ is the scaling factor for the homothety h_i; lastly, an element $x \in \mathcal{F}$ is $\mathcal{T}$-elliptic if and only if x has a conjugate in $\mathcal{G}_{n+1}$. The real $\mathcal{F}$-pretrees $\mathcal{T}$ are the limit pretrees for $(\mathcal{Y}_i)_{i=1}^{n}$ and the $\mathcal{F}$-pseudoforest $(\mathcal{T}, (\delta_i)_{i=1}^{n})$ is the limit pseudoforest for $(\mathcal{Y}_i, \delta_i)_{i=1}^{n}$. To summarize,

Theorem III.1 (cf. [Mut21b, Theorem III.3])**.** *Let $\psi \colon \mathcal{F} \to \mathcal{F}$ be an exponentially growing automorphism. Then there is:*

1. *a minimal $\mathcal{F}$-pseudoforest $(\mathcal{T}, (\delta_i)_{i=1}^{n})$ with trivial arc stabilizers;*
2. *a unique ψ-equivariant expanding homothety f of $(\mathcal{T}, (\delta_i)_{i=1}^{n})$; and*
3. *an element $x \in \mathcal{F}$ is $\mathcal{T}$-elliptic if and only if x $[\psi]$-grows polynomially.* □

Without metrics, there is not much one can do to compare limit pretrees. On the other hand, we do not expect limit pseudoforest to be well-defined (up to homothety) for a given outer automorphism — this is equivalent to existence of a canonical descending sequence of limit forests. The new idea is to pick a limit pseudoforest $(\mathcal{T}, (\delta_i)_{i=1}^n)$ and *normalize* its hierarchy $(\delta_i)_{i=1}^n$ using the *attracting laminations* for $[\psi]$. For the normalized hierarchy, the associated top level forest will be a *topmost limit forest* for $[\psi]$ related to the *topmost* attracting laminations. Once we have metrics to work with, we can prove that the topmost limit forests are essentially independent of the initial limit pseudoforest $(\mathcal{T}, (\delta_i)_{i=1}^n)$.

III.2 Attracting laminations

Fix an exponentially growing automorphism $\psi\colon \mathcal{F} \to \mathcal{F}$ with a descending sequence $(\mathcal{Y}_i, \delta_i)_{i=1}^n$ of limit forests. Each limit forest $(\mathcal{Y}_i, \delta_i)$ has matching stable laminations $\mathcal{L}^+_{\mathcal{Z}_i}[\psi_i]$ for $[\psi_i]$ rel. $\mathcal{Z}_i$, where $\mathcal{Z}_i$ is a $[\psi_i]$-invariant proper free factor system of $\mathcal{G}_i$. By Claim I.5, $\mathbb{R}(\mathcal{G}_i, \mathcal{Z}_i)$ is canonically identified with a subspace of $\mathbb{R}(\mathcal{G}_i)$ via a lifting map. As $\mathcal{G}_{i+1}$ is a malnormal subgroup system of $\mathcal{G}_i$, the space of lines $\mathbb{R}(\mathcal{G}_{i+1})$ is canonically identified with a closed subspace of $\mathbb{R}(\mathcal{G}_i)$ (exercise). By transitivity, $\mathbb{R}(\mathcal{G}_n) \subset \mathbb{R}(\mathcal{G}_{n-1}) \subset \cdots \subset \mathbb{R}(\mathcal{F})$.

An attracting lamination for $[\psi]$ in $\mathbb{R}(\mathcal{F})$ is the closure of a birecurrent line in $\mathbb{R}(\mathcal{F})$ with a ψ_*^k-attracting neighbourhood for some $k \geq 1$. The set of all attracting laminations for $[\psi]$ is *canonical* as it is defined using canonical constructs: $\mathbb{R}(\mathcal{F})$ and the homeomorphism $\psi_*\colon \mathbb{R}(\mathcal{F}) \to \mathbb{R}(\mathcal{F})$. Note that ψ_* permutes the attracting laminations for $[\psi]$.

Consider this chain of canonical embeddings: $\mathbb{R}(\mathcal{G}_i, \mathcal{Z}_i) \subset \mathbb{R}(\mathcal{G}_i) \subset \mathbb{R}(\mathcal{F})$. Quasiperiodicity is not preserved by the first embedding but a weaker form of it is. A line $[l]$ is birecurrent in an $\mathcal{F}$-forest if any closed interval $I \subset l$ has infinitely many translates contained in both ends of l; quasiperiodic lines are birecurrent.

Lemma III.2 (cf. [BFH00, Lemma 3.1.4])**.** *Let $f\colon (\mathcal{T}, d) \to (\mathcal{Y}, \delta)$ be an equivariant PL-map. A line is birecurrent in $\mathbb{R}(\mathcal{Y}, \delta)$ if and only if it is birecurrent in $\mathbb{R}(\mathcal{T})$.* (exercise) □

So leaves of $\mathcal{L}^+_{\mathcal{Z}_i}[\psi_i]$ are birecurrent in $\mathbb{R}(\mathcal{G}_i)$ and hence $\mathbb{R}(\mathcal{F})$; moreover, a ψ_{i*}^k-attracting neighbourhood of a line in $\mathbb{R}(\mathcal{G}_i, \mathcal{Z}_i)$ will lift to a ψ_*^k-attracting neighbourhood of the same line in $\mathbb{R}(\mathcal{F})$. (exercise) Thus the closure in $\mathbb{R}(\mathcal{F})$ of a stable lamination for $[\psi_i]$ rel. $\mathcal{Z}_i$, i.e. a component of $\mathcal{L}^+_{\mathcal{Z}_i}[\psi_i]$, is an attracting lamination for $[\psi]$.

Remark. This definition of "attracting" laminations is from [BFH00, Definition 3.1.5]. Shortly, we will define "topmost" attracting laminations as done in [BFH00, Section 6].

Lemma III.3 (cf. [BFH00, Lemma 3.1.10])**.** *Let $\psi\colon \mathcal{F} \to \mathcal{F}$ be an exponentially growing automorphism with a descending sequence $(\mathcal{Y}_i, \delta_i)_{i=1}^n$ of limit forests. The components of stable laminations $\mathcal{L}^+_{\mathcal{Z}_i}[\psi_i]$ $(1 \leq i \leq n)$ determine all the attracting laminations for $[\psi]$.*

Sketch of proof. Suppose that $[l] \in \mathbb{R}(\mathcal{F})$ is a birecurrent line with a ψ_*^k-attracting neighbourhood for some $k \geq 1$. If $\mathcal{G}_{n+1} \neq \emptyset$, then either it consists of only cyclic components or the restriction of ψ_n to $\mathcal{G}_{n+1}$ is polynomially growing. Either way, $\mathcal{G}_{n+1}$ cannot support an attracting lamination of ψ_n. Let $i \leq n$ be the maximal index for which $\mathbb{R}(\mathcal{G}_i) \subset \mathbb{R}(\mathcal{F})$ contains $[l]$. Birecurrence in $\mathbb{R}(\mathcal{F})$ and Lemma III.2 imply $[l]$ is birecurrent in $\mathbb{R}(\mathcal{G}_i, \mathcal{Z}_i)$ with a ψ_{i*}^k-attracting neighbourhood for some $i \leq n$ and $k \geq 1$. Following the proof of Claim II.8, assume some descending chain $(\mathcal{F}_{i,j})_{j=2}^{n_i}$ of proper free factor systems of $\mathcal{F}_{i,1} \coloneqq \mathcal{G}_i$ was used to construct $(\mathcal{Y}_i, \delta_i)$; then any birecurrent line in $\mathbb{R}(\mathcal{G}_i, \mathcal{Z}_i)$ with a ψ_{i*}^k-attracting neighbourhood is in $\mathbb{R}(\mathcal{F}_{i,n_i}, \mathcal{Z}_i)$. The proof of Lemma II.5 implies $[l] \in \mathcal{L}^+_{\mathcal{Z}_i}[\psi_i]$. □

The partially ordered (by inclusion) finite set of all attracting laminations for $[\psi]$ is canonical (by definition) and partially ordered by inclusion; an attracting lamination for $[\psi]$ is <u>topmost</u> if it is maximal in this poset. By Lemma II.5, ψ_{i*} transitively permutes the components of $\mathcal{L}^+_{\mathcal{Z}_i}[\psi_i]$; so the closure in $\mathbb{R}(\mathcal{F})$ of $\mathcal{L}^+_{\mathcal{Z}_i}[\psi_i] \subset \mathbb{R}(\mathcal{G}_i, \mathcal{Z}_i)$ is a ψ_*-orbit $\mathcal{L}^+_i[\psi]$ of attracting laminations for $[\psi]$. The goal is to *normalize* any limit pseudoforest $(\mathcal{T}, (d_i)_{i=1}^n)$ so that the top level is related to the topmost attracting laminations.

The next proposition is a repackaging of Theorem II.10 in the language of this chapter:

Proposition III.4. *Let $\psi\colon \mathcal{F} \to \mathcal{F}$ be an exponentially growing automorphism with a limit pseudoforest $(\mathcal{T}, (\delta_i)_{i=1}^n)$.*

For a nontrivial element $x \in \mathcal{F}$, the following statements are equivalent:

1. *the element x is $\mathcal{T}$-loxodromic;*
2. *the element x $[\psi]$-grows exponentially; and*
3. *the axis for x in $\mathbb{R}(\mathcal{F})$ weakly ψ_*-limits to an attracting lamination.*

Proof. The equivalence between Conditions 1–2 is part of Theorem III.1. Suppose $x \in \mathcal{F}$ is $\mathcal{T}$-loxodromic and the limit pseudoforest $(\mathcal{T}, (\delta_i)_{i=1}^n)$ is constructed from the descending sequence of limit forests $(\mathcal{Y}_i, \delta_i)$ for $1 \leq i \leq n$. By construction, the element x is conjugate to a $\mathcal{Y}_i$-loxodromic element $y \in \mathcal{G}_i$ for some $i \leq n$; in particular, x and y have the same axis in $\mathbb{R}(\mathcal{G}_i) \subset \mathbb{R}(\mathcal{F})$. The axis for y in $\mathbb{R}(\mathcal{G}_i, \mathcal{Z}_i) \subset \mathbb{R}(\mathcal{G}_i)$ weakly ψ_{i*}-limits to the stable laminations $\mathcal{L}^+_{\mathcal{Z}_i}[\psi_i] \subset \mathbb{R}(\mathcal{G}_i, \mathcal{Z}_i)$ by Theorem II.10; therefore, the shared axis for y and x in $\mathbb{R}(\mathcal{F})$ weakly ψ_*-limits to the attracting laminations for $[\psi]$ determined by $\mathcal{L}^+_{\mathcal{Z}_i}[\psi_i]$, i.e. the closure of $\mathcal{L}^+_{\mathcal{Z}_i}[\psi_i]$ in $\mathbb{R}(\mathcal{F})$.

Conversely, suppose $x \in \mathcal{F}$ is $\mathcal{T}$-elliptic. Then x is must be conjugate to a $\mathcal{Y}_n$-elliptic element $y \in \mathcal{G}_n$. If y is conjugate to an element of $\mathcal{Z}_i$, then the shared axis for y and x in the closed subspace $\mathbb{R}(\mathcal{Z}_i) \subset \mathbb{R}(\mathcal{F})$ cannot weakly ψ_*-limit to the attracting lamination for $[\psi]$ determined by a component of $\mathcal{L}^+_{\mathcal{Z}_i}[\psi_i]$ — such an attracting lamination contains lines not in $\mathbb{R}(\mathcal{Z}_i)$. If y is not conjugate to an element of $\mathcal{Z}_i$, then the axis for y in $\mathbb{R}(\mathcal{G}_i, \mathcal{Z}_i)$ does not weakly ψ_{i*}-limit to $\mathcal{L}^+_{\mathcal{Z}_i}[\psi_i]$ by Theorem II.10; therefore, the shared axis for y and x in $\mathbb{R}(\mathcal{F})$ cannot weakly ψ_*-limit to the attracting lamination for $[\psi]$ determined by a component of $\mathcal{L}^+_{\mathcal{Z}_i}[\psi_i]$. By Lemma III.3, we have exhausted all possibilities when $1 \leq i \leq n$ and the axis for x in $\mathbb{R}(\mathcal{F})$ cannot weakly ψ_*-limit to an attracting lamination for $[\psi]$. □

III.3 Pseudolaminations

Fix an exponentially growing automorphism $\psi\colon \mathcal{F} \to \mathcal{F}$ with a descending sequence $(\mathcal{Y}_i, \delta_i)_{i=1}^n$ of limit forests and let $(\mathcal{T}, (\delta_i)_{i=1}^n)$ be the limit pseudoforest for $(\mathcal{Y}_i, \delta_i)_{i=1}^n$. For $1 \le i \le n$, the stable laminations $\mathcal{L}^+_{\mathcal{Z}_i}[\psi_i]$ are contained in $\mathbb{R}(\mathcal{Y}_i, \delta_i) \subset \mathbb{R}(\mathcal{G}_i, \mathcal{Z}_i)$, where $\mathcal{Z}_i$ is some proper free factor system of $\mathcal{G}_i$.

Recall that $\mathcal{G}_1 = \mathcal{F}$ and $\mathcal{G}_{i+1} = \mathcal{G}[\mathcal{Y}_i]$ for $i \ge 1$. Let $\mathcal{T}_i \subset \mathcal{T}$ be the characteristic convex subsets for $\mathcal{G}_i$. By construction of $(\mathcal{T}, (\delta_i)_{i=1}^n)$, δ_i restricts to a $\mathcal{G}_i$-invariant convex pseudometric on $\mathcal{T}_i$ whose associated $\mathcal{G}_i$-forest can be equivariantly identified with $(\mathcal{Y}_i, \delta_i)$. Fix such an identification and let $\kappa_i\colon \mathcal{T}_i \to \mathcal{Y}_i$ denote the natural equivariant collapse map. The stable laminations $\mathcal{L}^+_{\mathcal{Z}_i}[\psi_i]$ are in $\mathbb{R}(\mathcal{Y}_i, \delta_i) \subset \mathbb{R}(\mathcal{G}_i, \mathcal{Z}_i)$; their leaves have unique lifts (via κ_i) to $\mathcal{T}_i \subset \mathcal{T}$; we call these pseudoleaves of $\mathcal{L}^+_{\mathcal{T}}[\psi_i]$. A pseudoleaf segment (of $\mathcal{L}^+_{\mathcal{T}}[\psi_i]$) is a closed interval in a (representative of a) pseudoleaf with nondegenerate κ_i-image in $\mathcal{Y}_i$.

Remarkably, the pseudoleaf segments detect weak ψ_*-limits of elements in attracting laminations. Let $\mathcal{L}^+_i[\psi]$ be the attracting laminations for $[\psi]$ determined by $\mathcal{L}^+_{\mathcal{Z}_i}[\psi_i]$, i.e. the closure in $\mathbb{R}(\mathcal{F})$ of the stable laminations $\mathcal{L}^+_{\mathcal{Z}_i}[\psi_i]$.

Proposition III.5. *Let $\psi\colon \mathcal{F} \to \mathcal{F}$ be an exponentially growing automorphism with a limit pseudoforest $(\mathcal{T}, (\delta_i)_{i=1}^n)$. For $1 \le j \le n$ and $\mathcal{T}$-loxodromic $x \in \mathcal{F}$, the axis for x in $\mathcal{T}$ contains a pseudoleaf segment of $\mathcal{L}^+_{\mathcal{T}}[\psi_j]$ if and only if the axis for x in $\mathbb{R}(\mathcal{F})$ weakly ψ_*-limits to the attracting laminations $\mathcal{L}^+_j[\psi]$.*

Proof. Let $(\mathcal{T}, (\delta_i)_{i=1}^n)$ be the limit pseudoforest for a descending sequence $(\mathcal{Y}_i, \delta_i)_{i=1}^n$ of limit forests for $[\psi]$. For $i \le n$, suppose $\tau_i^\circ\colon (\mathcal{T}_i^\circ, d_i^\circ) \to (\mathcal{T}_i^\circ, d_i^\circ)$ is the ψ_i-equivariant λ_i-Lipschitz PL-map induced by the descending sequence $(\tau_{i,j})_{j=1}^{n_i}$ of irreducible train tracks used to construct the limit forest $(\mathcal{Y}_i, \delta_i)$. Choose a metric free splitting $(\mathcal{T}^\star, d^\star)$ of $\mathcal{F}$ with trivial free factor system $\mathcal{F}[\mathcal{T}^\star]$ and whose characteristic subforest $(\mathcal{T}^\star(\mathcal{Z}_{i-1}), d^\star)$ is a metric blow-up of $(\mathcal{T}_i^\circ(\mathcal{Z}_{i-1}), d_i^\circ)$ for $i \le n$. In particular, $(\mathcal{T}^\star, d^\star)$ is a choice of coordinates for $\mathbb{R}(\mathcal{F})$ where the metric $d^\star$ is a sum of *independent* $\mathcal{F}$-invariant convex pseudometrics d_i° on the $\mathcal{F}$-orbits of $\mathcal{T}^\star(\mathcal{Z}_{i-1})$ and an arbitrary $\mathcal{F}$-invariant convex metric defined on the $\mathcal{F}$-orbit of $\mathcal{T}^\star(\mathcal{Z}_n)$. Pick some equivariant PL-map $\rho_i\colon (\mathcal{T}^\star(\mathcal{G}_i), d^\star) \to (\mathcal{T}_i^\circ, d_i^\circ)$. As the $\mathcal{G}_i$-orbit of $\mathcal{T}_i^\circ(\mathcal{Z}_{i-1})$ is τ_i°-invariant, the maps $(\tau_i^\circ)_{i=1}^n$ induce a ψ-equivariant PL-map $\tau^\star$ on $(\mathcal{T}^\star, d^\star)$.

Let $x \in \mathcal{F}$ be a $\mathcal{T}$-loxodromic element. By construction, the element x is conjugate to a $\mathcal{Y}_i$-loxodromic $y_i \in \mathcal{G}_i$ for some $i \le n$; let l_i° be the $\mathcal{T}_i^\circ$-axis for y_i. If $j = i$, then the equivalence in the propositions's statement follows from Theorem II.10. For the rest of the proof, we prove the equivalence when $j > i$. As we are going to invoke the same argument in the next proof, we mostly forget that l_i° is a $\mathcal{T}_i^\circ$-axis for a $\mathcal{Y}_i$-loxodromic element and only use the fact $[l_i^\circ] \in \mathbb{R}(\mathcal{Y}_i, \delta_i)$, i.e. l_i° projects to a line γ_i in $(\mathcal{Y}_i, \delta_i)$.

Suppose the $\mathcal{T}$-axis for x contains a pseudoleaf segment of $\mathcal{L}^+_{\mathcal{T}}[\psi_j]$ for some $j > i$. Then the $\mathcal{T}$-axis for y_i contains a pseudoleaf segment σ_j of $\mathcal{L}^+_{\mathcal{T}}[\psi_j]$ and $\kappa_i(\sigma_j)$ is a point

$\circ_i \in \gamma_i$ with nontrivial point stabilizer $G_{\circ_i} \coloneqq \mathrm{Stab}_{\mathcal{G}_i}(\circ_i)$. In Subsection II.2.3, we describe how the line γ_i in $(\mathcal{Y}_i, \delta_i)$ and point $\circ_i \in \gamma_i$ determine an algebraic iterated turn $(\epsilon, s_{i+1,1}^{-1} s_{i+1,2}; \varphi_{i+1})_{\mathcal{G}_{i+1}}$. Any iterated turn $(\beta_{i+1,m})_{m\geq 0}$ over $\mathcal{T}_{i+1}^\circ$ realizing this algebraic iterated turn limits to an interval $[\star_{i+1,1}, \star_{i+1,2}]$ in the metric completion $(\overline{\mathcal{Y}}_{i+1}, \delta_{i+1})$ by Theorem II.9.

If $j = i+1$, then $[\star_{i+1,1}, \star_{i+1,2}] \supset \kappa_{i+1}(\sigma_j)$ is not degenerate and $(\beta_{i+1,m})_{m\geq 0}$ weakly limits to a component of $\mathcal{L}^+_{\mathcal{Z}_{i+1}}[\psi_{i+1}]$ by Theorem II.9. Otherwise, we proceed by induction: assume $\kappa_{j-1}(\sigma_j)$ is a point $\circ_{j-1}$ in the interval $[\star_{j-1,1}, \star_{j-1,2}] \subset \overline{\mathcal{Y}}_{j-1}$ corresponding to the algebraic iterated turn $(\epsilon, s_{j-1,1}^{-1} s_{j-1,2}; \varphi_{j-1})_{\mathcal{G}_{j-1}}$, where $\circ_{j-1}$ has nontrivial stabilizer $G_{\circ_{j-1}}$. By the discussion in Subsection II.2.4 (and remark after Theorem II.9), the algebraic iterated turn over $\mathcal{G}_j$ and point $\circ_{j-1}$ in $[\star_{j-1,1}, \star_{j-1,2}]$ determine an algebraic iterated turn $(\epsilon, s_{j,1}^{-1} s_{j,2}; \varphi_j)_{\mathcal{G}_j}$ that limits to $[\star_{j,1}, \star_{j,2}] \subset \overline{\mathcal{Y}}_j$. Since $[\star_{j,1}, \star_{j,2}] \supset \kappa_j(\sigma_j)$ is not degenerate, any realization $(\beta_{j,m})_{m\geq 0}$ over $\mathcal{T}_j^\circ$ of this algebraic iterated turn weakly limits to a component of $\mathcal{L}^+_{\mathcal{Z}_j}[\psi_j]$ by Theorem II.9.

In either case ($j \geq i+1$), any realization over $\mathcal{T}^\star$ of $(\epsilon, s_{j,1}^{-1} s_{j,2}; \varphi_j)_{\mathcal{G}_j}$ weakly limits to (the closure in $\mathbb{R}(\mathcal{F})$ of) a component of $\mathcal{L}^+_{\mathcal{Z}_j}[\psi_j]$ (bounded cancellation). If $j > i+1$, any realization over $\mathcal{T}^\star$ of $(\epsilon, s_{i+1,1}^{-1} s_{i+1,2}; \varphi_{i+1})_{\mathcal{G}_{i+1}}$ weakly limits to a component of $\mathcal{L}^+_{\mathcal{Z}_j}[\psi_j]$ by transitivity. Hence the shared axis for y_i and x in $\mathbb{R}(\mathcal{F})$ weakly ψ_*-limits to a component of $\mathcal{L}^+_{\mathcal{Z}_j}[\psi_j]$. As $\psi_{j*} \colon \mathbb{R}(\mathcal{G}_j, \mathcal{Z}_j) \to \mathbb{R}(\mathcal{G}_j, \mathcal{Z}_j)$ acts transitively on the components of $\mathcal{L}^+_{\mathcal{Z}_j}[\psi_j]$, the axis for x in $\mathbb{R}(\mathcal{F})$ weakly ψ_*-limits to $\mathcal{L}^+_j[\psi]$, the closure in $\mathbb{R}(\mathcal{F})$ of $\mathcal{L}^+_{\mathcal{Z}_j}[\psi_j]$.

Conversely, suppose the axis $[l^\star]$ for y_i (and x) in $\mathbb{R}(\mathcal{F})$ weakly ψ_*-limits to $\mathcal{L}^+_j[\psi]$ for some $j > i$. Using $(\mathcal{T}^\star, d^\star)$-coordinates, the axis $\tau_*^{\star m}(l^\star)$ contains arbitrarily d_j°-long leaf segments of $\mathcal{L}^+_j[\psi]$ for $m \gg 1$. So $\tau_*^{\star M}(l^\star)$ has a $\mathcal{L}^+_j[\psi]$-leaf segment $I^\star \subset \mathcal{T}^\star(\mathcal{Z}_{j-1})$ with d_j°-length $L > C' \coloneqq \frac{2C[\tau^\star]}{\lambda_j - 1}$ for $M \gg 1$. As τ_j° is a train track, $\tau_*^{\star m}(l^\star)$ has a $\mathcal{L}^+_j[\psi]$-leaf segment surviving from $I^\star$ with d_j°-length $> \lambda_j^{M-m}(L - C')$ for $m \geq M$.

The ρ_i-image of $I^\star \subset \tau_*^{\star M}(l^\star)$ is a vertex $v \in \tau_{i*}^{\circ M}(l_i^\circ)$ with nontrivial stabilizer. Since a nondegenerate part of $I^\star$ survives in $\psi_*^m(l^\star)$ for all $m \geq M$, we have $\tau_i^{\circ(m-M)}(v) \in \tau_{i*}^{\circ m}(l_i^\circ)$ for all $m \geq M$ and $h_i^{-M}(\pi_i^\circ(v)) \in \gamma_i$ has a nontrivial stabilizer $G_v \coloneqq \mathrm{Stab}_{\mathcal{G}_i}(h_i^{-M}(\pi_i^\circ(v)))$, where h_i is the ψ_i-equivariant λ_i-homothety on $(\mathcal{Y}_i, \delta_i)$. As before, the line γ_i, point $h_i^{-M}(\pi_i^\circ(v)) \in \gamma_i$, and equivariant PL-maps $\rho_{i+1}, \ldots, \rho_j$ determine nested iterated turns over $\mathcal{T}_{i+1}^\circ, \ldots, \mathcal{T}_j^\circ$ limiting to intervals in $\overline{\mathcal{Y}}_{i+1}, \ldots, \overline{\mathcal{Y}}_j$. By the computation in the previous paragraph and quasiperiodicity of stable laminations, the last iterated turn over $\mathcal{T}_j^\circ$ weakly limits to a component of $\mathcal{L}^+_{\mathcal{Z}_j}[\psi_j]$. So the corresponding interval $[\star_{j,1}, \star_{j,2}] \subset \overline{\mathcal{Y}}_j$ is not degenerate (Theorem II.9) and the $\mathcal{T}$-axis for y_i has an intersection with $\mathcal{T}_j$ whose κ_j-image is $[\star_{j,1}, \star_{j,2}]$. By the description of intervals in $\mathcal{Y}_j$, $[\star_{j,1}, \star_{j,2}]$ contains a leaf segment of $\pi_{j*}^\circ(\mathcal{L}^+_{\mathcal{Z}_j}[\psi_j])$; therefore, the $\mathcal{T}$-axes of y_i and x contain pseudoleaf segments of $\mathcal{L}^+_{\mathcal{T}}[\psi_j]$. □

In fact, the containment relation on pseudoleaf segments detects the partial order on the set of attracting laminations:

Claim III.6. *For $1 \le i < j \le n$, a pseudoleaf segment of $\mathcal{L}^+_{\mathcal{T}}[\psi_i]$ contains a pseudoleaf segment of $\mathcal{L}^+_{\mathcal{T}}[\psi_j]$ if and only if $\mathcal{L}^+_i[\psi]$ contains $\mathcal{L}^+_j[\psi]$.*

We only sketch the proof as it is almost identical to the proof of Proposition III.5.

Sketch of proof. Fix $j > i$ then let $[l_i^\circ]$ be an eigenline in $(\mathcal{T}_i^\circ, d_i^\circ)$ of $[\tau_i^{\circ k}]$ for some $k \ge 1$, and $l^\star$ be the lift of l_i° to $(\mathcal{T}^\star, d^\star)$. The projection γ_i (of l_i°) is a line in $(\mathcal{Y}_i, \delta_i)$ and we denote by $l_i \subset \mathcal{T}_i$ its lift via κ_i to a pseudoleaf of $\mathcal{L}^+_{\mathcal{T}}[\psi_i]$.

Suppose a pseudoleaf segment of $\mathcal{L}^+_{\mathcal{T}}[\psi_i]$ contains a pseudoleaf segment of $\mathcal{L}^+_{\mathcal{T}}[\psi_j]$. Then the pseudoleaf l_i contains a pseudoleaf segment σ_j of $\mathcal{L}^+_{\mathcal{T}}[\psi_j]$ and $\kappa_i(\sigma_j)$ is a point $\circ_i \in \gamma_i$ with nontrivial point stabilizer $G_{\circ_i}$. By the same argument as in the previous proof, the line $l^\star$ in $\mathbb{R}(\mathcal{F})$ weakly ψ_*-limits to $\mathcal{L}^+_j[\psi]$. Note that $\psi_*^k[l^\star] = [l^\star]$ in $\mathbb{R}(\mathcal{F})$ as $[l_i^\circ]$ is an eigenline for $[\tau_i^{\circ k}]$; moreover, $\mathcal{L}^+_i[\psi]$ consists of the closures in $\mathbb{R}(\mathcal{F})$ of $[l^\star]$, …, $\psi_*^{k-1}[l^\star]$ since $\mathcal{L}^+_i[\psi]$ is a ψ_*-orbit of attracting laminations. So $\mathcal{L}^+_i[\psi] \supset \mathcal{L}^+_j[\psi]$.

Conversely, suppose $\mathcal{L}^+_i[\psi] \supset \mathcal{L}^+_j[\psi]$. As $\mathcal{L}^+_i[\psi]$ and $\mathcal{L}^+_j[\psi]$ are ψ_*-orbits of attracting laminations, the line $l^\star$ contains arbitrarily d_j°-long leaf segments of $\mathcal{L}^+_j[\psi]$. By the same argument as in the previous proof, the pseudoleaf l_i, and hence some pseudoleaf segment of $\mathcal{L}^+_{\mathcal{T}}[\psi_i]$, contains a pseudoleaf segment of $\mathcal{L}^+_{\mathcal{T}}[\psi_j]$. □

III.4 Existence of topmost limit forests

Fix an exponentially growing automorphism $\psi\colon \mathcal{F} \to \mathcal{F}$ with a descending sequence $(\mathcal{Y}_i, \delta_i)_{i=1}^n$ of limit forests and let $(\mathcal{T}, (\delta_i)_{i=1}^n)$ be the limit pseudoforest for $(\mathcal{Y}_i, \delta_i)_{i=1}^n$. Each limit forest $(\mathcal{Y}_i, \delta_i)$ has stable laminations $\mathcal{L}^+_{\mathcal{Z}_i}[\psi_i]$ for $[\psi_i]$ rel. $\mathcal{Z}_i$. Let $\mathcal{L}^+_{\mathcal{T}}[\psi_i]$ be the lifts to $\mathcal{T}$ of leaves in $\mathcal{L}^+_{\mathcal{Z}_i}[\psi_i]$, $\mathcal{L}^+_i[\psi]$ the closure in $\mathbb{R}(\mathcal{F})$ of $\mathcal{L}^+_{\mathcal{Z}_i}[\psi_i]$, and $(\mathcal{A}_j^{top}[\psi_i])_{j=1}^{k_i}$ the subsequence of $(\mathcal{L}^+_j[\psi])_{j=i}^n$ consisting of topmost attracting laminations for $[\psi_i]$. So $\mathcal{A}_j^{top}[\psi_i] = \mathcal{L}^+_{\iota(i,j)}[\psi]$ for some subsequence $(\iota(i,j))_{j=1}^{k_i}$ of $(j)_{j=i}^n$ with $\iota(i,1) = i$ and, if $i \ge 2$ and $k_{i-1} \ge 2$, then $(\iota(i-1,j))_{j=2}^{k_{i-1}}$ is a subsequence of $(\iota(i,j))_{j=1}^{k_i}$.

The ψ_*-orbit $\mathcal{A}_1^{top}[\psi_n] = \mathcal{L}^+_n[\psi]$ contains attracting laminations for $[\psi_n]$ and we say the $\mathcal{G}_n$-invariant convex metric δ_n on $\mathcal{T}_n$ is the <u>normal form rel. $[\psi_n]$</u> for the 1-level hierarchy (δ_n). For the rest of the section, we inductively normalize the hierarchy.

III.4.1 First induction step

Consider the $\mathcal{G}_{n-1}$-pretree $\mathcal{T}_{n-1}$. If $k_{n-1} = 1$ (equivalently, $\mathcal{L}^+_{n-1}[\psi]$ contains $\mathcal{L}^+_n[\psi]$), then $\mathcal{A}_1^{top}[\psi_{n-1}] = \mathcal{L}^+_{n-1}[\psi]$ contains all attracting laminations for $[\psi_{n-1}]$ and the $\mathcal{G}_{n-1}$-invariant convex pseudometric $\delta_{\iota(n-1,1)} = \delta_{n-1}$ on $\mathcal{T}_{n-1}$ is the <u>normal form rel. $[\psi_{n-1}]$</u> for the 2-level hierarchy (δ_{n-1}, δ_n).

Otherwise, $k_{n-1} = 2$ and the union $\bigcup_{j=1}^{2} \mathcal{A}_j^{top}[\psi_{n-1}] = \mathcal{L}_{n-1}^{+}[\psi] \cup \mathcal{L}_n^{+}[\psi]$ contains all attracting laminations for $[\psi_{n-1}]$. This amounts to assuming $\kappa_{n-1} \colon \mathcal{T}_{n-1} \to \mathcal{Y}_{n-1}$ is injective on the pseudoleaves of $\mathcal{L}_{\mathcal{T}}^{+}[\psi_{n-1}]$ by Claim III.6. Suppose $\mathcal{F}_{n-1,1} \coloneqq \mathcal{G}_{n-1}, \ldots, \mathcal{F}_{n-1,m}$ were the proper free factor systems of $\mathcal{G}_{n-1}$ used to construct $(\mathcal{Y}_{n-1}, \delta_{n-1})$ and let $\mathcal{T}_{n-1,1}$, $\ldots$, $\mathcal{T}_{n-1,m}$ be their corresponding characteristic convex subsets in $\mathcal{T}_{n-1}$. By Lemma II.6, every closed interval in the characteristic convex subsets $\mathcal{Y}_{n-1}(\mathcal{F}_{n-1,m})$ is a finite concatenation of leaf segments of $\mathcal{L}_{\mathcal{Z}_{n-1}}^{+}[\psi_{n-1}]$.

Since κ_{n-1} is injective on the pseudoleaves of $\mathcal{L}_{\mathcal{T}}^{+}[\psi_{n-1}]$, every closed interval in $\mathcal{T}_{n-1,m}$ is a finite concatenation of pseudoleaf segments of $\mathcal{L}_{\mathcal{T}}^{+}[\psi_{n-1}]$ and closed intervals in $\widehat{\mathcal{T}}_n$, where $\widehat{\mathcal{T}}_n$ is the κ_{n-1}-preimage of the characteristic convex subsets $\mathcal{Y}_{n-1}(\mathcal{G}_n)$. Thus the $\mathcal{F}_n$-invariant convex metric δ_n on $\widehat{\mathcal{T}}_n$ extends to an $\mathcal{F}_{n-1,m}$-invariant convex pseudometric on $\mathcal{T}_{n-1,m}$. The sum $\delta_{n-1} \oplus \delta_n$ of δ_{n-1} and δ_n is a factored $\mathcal{F}_{n-1,m}$-invariant convex metric on $\mathcal{T}_{n-1,m}$. By our inductive description of intervals in $\mathcal{Y}_{n-1}$ (Lemma II.7), the convex pseudometric δ_n extends equivariantly to $\mathcal{T}_{n-1}$ as $\lambda_n > 1$. The factored $\mathcal{G}_{n-1}$-invariant convex metric $\oplus_{j=1}^{2} \delta_{\iota(n-1,j)} = \delta_{n-1} \oplus \delta_n$ on $\mathcal{T}_{n-1}$ is the normal form rel. $[\psi_{n-1}]$ for (δ_{n-1}, δ_n).

III.4.2 Remaining induction steps

Consider the $\mathcal{G}_{i-1}$-pretree $\mathcal{T}_{i-1}$ for $1 < i < n$. Assume the normal form rel. $[\psi_i]$ for $(\delta_j)_{j=i}^{n}$ is the *factored* $\mathcal{G}_i$-invariant convex pseudometric $\oplus_{j=1}^{k_i} \delta_{\iota(i,j)}$ on $\mathcal{T}_i$. If $k_{i-1} = 1$, then $\mathcal{A}_1^{top}[\psi_{i-1}]$ contains all attracting laminations for $[\psi_{i-1}]$ and the $\mathcal{G}_{i-1}$-invariant convex pseudometric $\delta_{\iota(i-1,1)} = \delta_{i-1}$ on $\mathcal{T}_{i-1}$ is the normal form rel. $[\psi_{i-1}]$ for $(\delta_j)_{j=i-1}^{n}$. Otherwise, the union $\bigcup_{j=1}^{k_{i-1}} \mathcal{A}_j^{top}[\psi_{i-1}]$ contains all attracting laminations for $[\psi_{i-1}]$. The rest is almost identical to the first induction step: by our description of intervals in $\mathcal{Y}_{i-1}$, the convex pseudometrics $\delta_{\iota(i-1,j)}$ $(j \geq 2)$ on $\widehat{\mathcal{T}}_i$ extend equivariantly to $\mathcal{T}_{i-1}$ since $\lambda_{\iota(i-1,j)} > 1$; the normal form rel. $[\psi_{i-1}]$ for $(\delta_j)_{j=i-1}^{n}$ is the factored $\mathcal{G}_{i-1}$-invariant convex pseudometric $\oplus_{j=1}^{k_{i-1}} \delta_{\iota(i-1,j)}$ on $\mathcal{T}_{i-1}$. By induction, the normal form rel. $[\psi]$ for $(\delta_i)_{i=1}^{n}$ is the factored $\mathcal{F}$-invariant convex pseudometric $\oplus_{j=1}^{k} \delta_{\iota(j)}$ on $\mathcal{T}$, where $k \coloneqq k_1$ and $\iota(j) \coloneqq \iota(1,j)$.

Let $(\mathcal{Y}, \oplus_{j=1}^{k} \delta_{\iota(j)})$ be the associated factored $\mathcal{F}$-forest. The real $\mathcal{F}$-pretrees $\mathcal{Y}$ are minimal and have trivial arc stabilizers since the pseudometric $\oplus_{j=1}^{k} \delta_{\iota(j)}$ on $\mathcal{T}$ is convex. The $(\lambda_i)_{i=1}^{n}$-homothety f induces a ψ-equivariant $\oplus_{j=1}^{k} \lambda_{\iota(j)}$-dilation on $(\mathcal{Y}, \oplus_{j=1}^{k} \delta_{\iota(j)})$: a $\lambda_{\iota(j)}$-homothety with respect to each $\mathcal{F}$-invariant convex pseudometric $\delta_{\iota(j)}$. By Proposition III.5, a nontrivial element of $\mathcal{F}$ is $\delta_{\iota(j)}$*-loxodromic* if and only if its axis in $\mathbb{R}(\mathcal{F})$ weakly ψ_*-limits to $\mathcal{A}_j^{top}[\psi_1]$ — here, "$\delta_{\iota(j)}$-loxodromic" means the element acts loxodromically on the associated $\mathcal{F}$-forest for $\delta_{\iota(j)}$. The factored $\mathcal{F}$-forest $(\mathcal{Y}, \oplus_{j=1}^{k} \delta_{\iota(j)})$ is the topmost limit forest for $(\mathcal{Y}_i, \delta_i)_{i=1}^{n}$. This proves the existence part of our main theorem:

Theorem III.7. *Let $\psi\colon \mathcal{F} \to \mathcal{F}$ be an exponentially growing automorphism of a free group system and $(\mathcal{A}_j^{top}[\psi])_{j=1}^k$ the ψ_*-orbits of topmost attracting laminations for $[\psi]$.*

Then there is:

1. *a minimal factored $\mathcal{F}$-forest $(\mathcal{Y}, \oplus_{j=1}^k \delta_j)$ with trivial arc stabilizers;*
2. *a unique ψ-equivariant expanding dilation $h\colon (\mathcal{Y}, \oplus_{j=1}^k \delta_j) \to (\mathcal{Y}, \oplus_{j=1}^k \delta_j)$; and*
3. *for $1 \le j \le k$, a nontrivial element $x \in \mathcal{F}$ is δ_j-loxodromic if and only if its axis in $\mathbb{R}(\mathcal{F})$ weakly ψ_*-limits to $\mathcal{A}_j^{top}[\psi]$.* □

In the next section, we prove topmost limit forests are unique up to equivariant dilation!

III.5 Uniqueness of topmost limit forest

With the same techniques, we can now prove a special case of our main uniqueness result:

Proposition III.8. *Let $\psi\colon \mathcal{F} \to \mathcal{F}$ be an exponentially growing automorphism. Any two topmost limit forests for $[\psi]$ have unique equivariant dilations between them.*

Thus Theorem III.7 produces *the* topmost limit forest for $[\psi]$; consequently, its equivariant dilation invariants, e.g. the Gaboriau–Levitt index, are now invariants of $[\psi]$.

Sketch of proof. The strategy is to reconstruct, up to rescaling, one limit pseudoforest for $[\psi]$ by "shuffling" any other given limit pseudoforest for $[\psi]$.

Let $(\mathcal{T}, (\delta_i)_{i=1}^n)$ and $(\mathcal{T}', (\delta_i')_{i=1}^{n'})$ be the limit pseudoforests for descending sequences $(\mathcal{Y}_i, \delta_i)_{i=1}^n$ and $(\mathcal{Y}_i', \delta_i')_{i=1}^{n'}$ of limit forests for $[\psi]$ respectively. Note that $n = n'$ is the number of ψ_*-orbits of attracting laminations for $[\psi]$. Let $\mathcal{L}_i^+[\psi]$ be the closure in $\mathbb{R}(\mathcal{F})$ of $\mathcal{L}_{\mathcal{Z}_i}^+[\psi_i] \subset \mathbb{R}(\mathcal{G}_i, \mathcal{Z}_i)$ for $1 \le i \le n$ — here the sequence $(\psi_i\colon \mathcal{G}_i \to \mathcal{G}_i)_{i=1}^n$ of automorphisms corresponds to the first descending sequence $(\mathcal{Y}_i, \delta_i)_{i=1}^n$. Then $\mathcal{L}_{\sigma\cdot i}^+[\psi]$ is the closure in $\mathbb{R}(\mathcal{F})$ of $\mathcal{L}_{\mathcal{Z}_i'}^+[\psi_i'] \subset \mathbb{R}(\mathcal{G}_i', \mathcal{Z}_i')$ for $1 \le i \le n$, where $(\psi_i')_{i=1}^n$ corresponds to $(\mathcal{Y}_i', \delta_i')_{i=1}^n$ and σ is a permutation of $\{1, \ldots, n\}$.

Since $\mathcal{L}_{\sigma\cdot 1}^+[\psi]$ is not in $\mathcal{L}_{\sigma\cdot j}^+[\psi]$ for $j > 1$, the convex pseudometric $\delta_{\sigma\cdot 1}$ equivariantly extend to $\mathcal{T}$ by the same argument used to construct topmost limit forests. If $\sigma \cdot 1 \ne 1$, then we have an $\mathcal{F}$-invariant $(n-1)$-level factored hierarchy $(\delta_1 \oplus \delta_{\sigma\cdot 1}, \ldots)$ on $\mathcal{T}$ and, by restricting the domain of δ_1, we get an $\mathcal{F}$-invariant n-level hierarchy $(\delta_{\sigma\cdot 1}, \delta_1, \ldots)$ on $\mathcal{T}$ with full support; moreover, the ψ-equivariant pretree-automorphism $h\colon \mathcal{T} \to \mathcal{T}$ is still an expanding homothety with respect to this new hierarchy — this the first "shuffle". Let $(\mathcal{Y}_1^\sigma, \delta_{\sigma\cdot 1})$ be the associated $\mathcal{F}$-forest for the convex pseudometric $\delta_{\sigma\cdot 1}$ on $\mathcal{T}$. A nontrivial element of $\mathcal{F}$ is $\delta_{\sigma\cdot 1}$-loxodromic if and only if its axis in $\mathbb{R}(\mathcal{F})$ weakly ψ_*-limits to $\mathcal{L}_{\sigma\cdot 1}^+[\psi]$, i.e. $\mathcal{G}[\mathcal{Y}_1^\sigma] = \mathcal{G}_2'$. In particular, $(\mathcal{Y}_1^\sigma, \delta_{\sigma\cdot 1})$ is an expanding forest for $[\psi]$ like $\mathcal{Y}_1'$. So $(\mathcal{Y}_1^\sigma, \delta_{\sigma\cdot 1})$ is equivariantly homothetic to $(\mathcal{Y}_1', \delta_1')$ by Corollary II.11.

We shuffle inductively. As $\mathcal{L}^+_{\sigma\cdot i}[\psi]$ is not contained in $\mathcal{L}^+_{\sigma\cdot j}[\psi]$ for $j > i$, the convex pseudometric $\delta_{\sigma\cdot i}$ equivariantly extends to $\mathcal{T}(\mathcal{G}'_i)$, the characteristic convex subsets of $\mathcal{T}$ for $\mathcal{G}'_i$. As before, we get an $\mathcal{F}$-invariant i-level hierarchy $(\delta_{\sigma\cdot j})^i_{j=1}$ on $\mathcal{T}$ and h is an expanding homothety with respect to this hierarchy. A nontrivial element of $\mathcal{G}'_i$ is $\delta_{\sigma\cdot i}$-loxodromic if and only if its axis in $\mathbb{R}(\mathcal{G}'_i)$ weakly ψ_*-limits to $\mathcal{L}^+_{\sigma\cdot i}[\psi]$. So the associated $\mathcal{G}'_i$-forest $(\mathcal{Y}^\sigma_i, \delta_{\sigma\cdot i})$ for the convex pseudometric $\delta_{\sigma\cdot i}$ on $\mathcal{T}(\mathcal{G}'_i)$ is equivariantly homothetic to $(\mathcal{Y}'_i, \delta'_i)$ by Corollary II.11.

By the n^{th} shuffle, we have an $\mathcal{F}$-invariant n-level hierarchy $(\delta_{\sigma\cdot i})^n_{i=1}$ on $\mathcal{T}$ with full support; moreover, the ψ-equivariant pretree-automorphism $h\colon \mathcal{T} \to \mathcal{T}$ is still an expanding homothety with respect to this new hierarchy. Uniqueness of the equivariant blow-up construction implies $(\mathcal{T}, (\delta_{\sigma\cdot i})^n_{i=1})$ is uniquely equivariantly homothetic to $(\mathcal{T}', (\delta'_i)^n_{i=1})$. Shuffling guarantees the normal forms rel. $[\psi]$ for $(\delta_i)^n_{i=1}$ and $(\delta_{\sigma\cdot i})^n_{i=1}$ (using the orderings $(\mathcal{L}^+_i[\psi])^n_{i=1}$ and $(\mathcal{L}^+_{\sigma\cdot i}[\psi])^n_{i=1}$ respectively) are the same factored convex pseudometrics on $\mathcal{T}$ up to a reordering (by σ) of the *factors* $\delta_{\iota(j)}$. So there is a unique equivariant dilation from the topmost limit forest for $(\mathcal{Y}_i, \delta_i)^n_{i=1}$ to the topmost limit forest for $(\mathcal{Y}'_i, \delta'_i)^n_{i=1}$. □

As a corollary of this proof, the underlying pretree of a limit pseudoforest for $[\psi]$ is unique, i.e. $[\psi]$ has a unique limit pretree:

Corollary III.9. *Let $\psi\colon \mathcal{F} \to \mathcal{F}$ be an automorphism. Any two limit pretrees for $[\psi]$ are equivariantly pretree-isomorphic.* □

As with the topmost limit forest, equivariant pretree invariants of the limit pretree, e.g. the Gaboriau–Levitt index as extended in [Mut21b, Appendix A], are invariants of $[\psi]$.

IV Topmost expanding forests

Let $\psi\colon \mathcal{F} \to \mathcal{F}$ be an exponentially growing automorphism. Topmost $[\psi]$-expanding forests are factored $\mathcal{F}$-forests satisfying the conclusion of Theorem III.7. Universality in our main theorem states that there is *essentially* only one topmost $[\psi]$-expanding forest.

Fix a descending sequence $(\mathcal{Y}_i, \delta_i)_{i=1}^n$ of limit forest for $[\psi]$. Set $\mathcal{G}_1 \coloneqq \mathcal{F}$ and $\mathcal{G}_{i+1} = \mathcal{G}[\mathcal{Y}_i]$ for $i \geq 1$; suppose $(\mathcal{Y}_i, \delta_i)$ is the limit forest for the descending sequence $(\tau_{i,j}\colon \mathcal{T}_{i,j} \to \mathcal{T}_{i,j})_{j=1}^{n_i}$ of irreducible train tracks for $[\psi_i]$ rel. $\mathcal{Z}_i$. Set $\mathcal{F}_{i,1} \coloneqq \mathcal{G}_i$ and $\mathcal{F}_{i,j+1} \coloneqq \mathcal{F}[\mathcal{T}_{i,j}]$ for $j \geq 1$; let $\mathcal{L}^+_{\mathcal{Z}_i}[\psi_i]$ be the stable laminations for $[\psi_i]$ in $\mathbb{R}(\mathcal{G}_i, \mathcal{Z}_i)$, $\mathcal{L}^+_i[\psi]$ the closure of $\mathcal{L}^+_{\mathcal{Z}_i}[\psi_i]$ in $\mathbb{R}(\mathcal{F})$, $(\mathcal{A}^{top}_j[\psi_i])_{j=1}^{k_i}$ the subsequence of $(\mathcal{L}^+_j[\psi])_{j=i}^n$ consisting of topmost attracting laminations for $[\psi_i]$, and $(\mathcal{Y}^*_i, \oplus_{j=1}^{k_i}\delta_{\iota(i,j)})$ the topmost limit forest for $(\mathcal{Y}_j, \delta_j)_{j=i}^n$. So $\mathcal{A}^{top}_j[\psi_i] = \mathcal{L}^+_{\iota(i,j)}[\psi]$ for some subsequence $(\iota(i,j))_{j=1}^{k_i}$ of $(j)_{j=i}^n$ with $\iota(i,1) = i$.

IV.1 Decomposing topmost limit forests

For our uniqueness theorem, we will have to generalize Lemmas V.3 and V.6 to topmost limit forests. This requires us to consider splittings with possibly noncyclic edge stabilizers.

In Section II.4, we described $(\mathcal{Y}_i, \delta_i)$ as an *iterated* metric blow-up of the free splittings $(\mathcal{T}_{i,j})_{j=1}^{n_i-1}$ and the $\mathcal{F}_{i,n_i}$-forest $(\mathcal{Y}_{i,n_i}, \delta_i)$ with respect to ψ_i-iteration, where the latter was the limit forest for $[\tau_{i,n_i}]$ and the edges of $\mathcal{T}_{i,j}$ were collapsed. For $1 \leq i < n$, let $\mathcal{Y}^*_i(\mathcal{F}_{i,n_i})$ be the characteristic convex subsets of $\mathcal{Y}^*_i$ for $\mathcal{F}_{i,n_i}$ and $\widehat{\mathcal{Y}}^*_i(\mathcal{G}_{i+1})$ the closure in $(\mathcal{Y}^*_i, \oplus_{j=1}^{k_i}\delta_{\iota(i,j)})$ of the characteristic convex subsets for $\mathcal{G}_{i+1}$. By Lemma II.6, intervals in $\mathcal{Y}^*_i(\mathcal{F}_{i,n_i})$ are finite concatenations of leaf segments of $\mathcal{L}^+_{\mathcal{Z}_i}[\psi_i]$ and closed intervals in $\widehat{\mathcal{Y}}^*_i(\mathcal{G}_{i+1})$.

Two lines in $(\mathcal{Y}^*_i(\mathcal{F}_{i,n_i}), \oplus_{j=1}^{k_i}\delta_{\iota(i,j)})$ representing leaves in $\mathcal{L}^+_{\mathcal{Z}_i}[\psi_i]$ *overlap* if they have a nondegenerate intersection; overlapping generates an equivalence relation and each *overlapping* class is identified with its union in $(\mathcal{Y}^*_i(\mathcal{F}_{i,n_i}), \oplus_{j=1}^{k_i}\delta_{\iota(i,j)})$. The overlapping classes $L^+_{\mathcal{Z}_i}$ and the $\mathcal{F}_{i,n_i}$-orbits of components of $\widehat{\mathcal{Y}}^*_i(\mathcal{G}_{i+1})$ form a $\mathcal{F}_{i,n_i}$-invariant *transverse covering* of $\mathcal{Y}^*_i(\mathcal{F}_{1,n_1})$ (see [Gui04, Definition 4.6]). Let $\mathcal{S}_i$ be a *bipartite* simplicial $\mathcal{F}_{i,n_i}$-pretree: one group of vertices ("component vertices") in equivariant bijective correspondence with the components of the transverse covering (overlapping classes $L^+_{\mathcal{Z}_i}$ and translates of components of $\widehat{\mathcal{Y}}^*_i(\mathcal{G}_{i+1})$); the other group of vertices ("intersection vertices") in equivariant bijective correspondence with the nonempty pairwise intersections of distinct components of the transverse covering; an edge connects two vertices in distinct groups if the intersection vertex corresponds to an intersection involving the component of the transverse covering corresponding to the "component vertex".

The forest $(\mathcal{Y}^*_i(\mathcal{F}_{i,n_i}), \oplus_{j=1}^{k_i}\delta_{\iota(i,j)})$ is a graph of actions with *skeleton* $\mathcal{S}_i$ and whose "vertex trees" are the components of the transverse covering [Gui04, Lemma 4.7]; note that $\mathcal{S}_i$ has finitely many $\mathcal{F}_{i,n_i}$-orbits of vertices as the number of $\mathcal{F}_{i,n_i}$-orbits of overlapping classes $L^+_{\mathcal{Z}_i}$ is at most the number of components in $\mathcal{L}^+_{\mathcal{Z}_i}[\psi_i]$. Let $\mathcal{H}_i[\psi]$ denote the subgroup system of finite type corresponding to the (setwise) stabilizers of overlapping classes $L^+_{\mathcal{Z}_i}$ — this subgroup system, called the support of $\mathcal{L}^+_i[\psi]$, is well-defined for the ψ_*-orbit $\mathcal{L}^+_i[\psi]$ of attracting laminations. Since the ψ_i-equivariant expanding homothety h_i on $(\mathcal{Y}^*_i, \oplus_{j=1}^{k_i} \delta_{\iota(i,j)})$ permutes the overlapping classes (and orbits of $\widehat{\mathcal{Y}}^*_i(\mathcal{G}_{i+1})$), it induces a simplicial automorphism $\sigma_i \colon \mathcal{S}_i \to \mathcal{S}_i$ that preserves the "type" of vertex trees. Thus the support $\mathcal{H}_i[\psi]$ is $[\psi_i]$-invariant.

Pick some $i \in \{\iota(1,j) : 1 \le j \le k_1\}$; so $\mathcal{L}^+_i[\psi]$ is a ψ_*-orbit of topmost attracting laminations for $[\psi]$. Set $(\mathcal{X}_i, \delta_i) \coloneqq (\mathcal{Y}_i, \delta_i)$, $f_i \coloneqq h_i$, and $\lambda \coloneqq \lambda[\tau_{i,n_i}]$. For $1 \le l < i$, assume the $\mathcal{G}_{l+1}$-forest $(\mathcal{X}_{l+1}, \delta_i)$ is the associated $\mathcal{G}_{l+1}$-forest for the pseudometric δ_i on $\mathcal{Y}^*_{l+1}$ and $f_{l+1} \colon (\mathcal{X}_{l+1}, \delta_i) \to (\mathcal{X}_{l+1}, \delta_i)$ a ψ_{l+1}-equivariant λ-homothety. Define $(\mathcal{X}_l, \delta_i)$ to be the iterated metric blow-up of the free splittings $(\mathcal{T}_{l,j})_{j=1}^{n_l-1}$, the skeleton $\mathcal{S}_l$, and $(\mathcal{X}_{l+1}, \delta_i)$ with respect to ψ_l-iteration. Specifically, let $(\mathcal{T}^\diamond_l, \delta^\diamond_i)$ be an equivariant metric blow-up of $(\mathcal{T}_{l,j})_{j=1}^{n_l-1}$, $\mathcal{S}_l$, and $(\mathcal{X}_{l+1}, \delta_i)$. When the metric δ_i is extended appropriately to $\delta^\diamond_i$, the simplicial automorphisms $(\tau_{l,j})_{j=1}^{n_l-1}$, σ_l and the homothety f_{l+1} induce a unique ψ_l-equivariant λ-Lipschitz map $\tau^\diamond_l \colon (\mathcal{T}^\diamond_l, \delta^\diamond_i) \to (\mathcal{T}^\diamond_l, \delta^\diamond_i)$ that linearly extends f_{l+1}. As with limit forests, we construct an equivariant metric surjection $\pi^\diamond_l \colon (\mathcal{T}^\diamond_l, \delta^\diamond_i) \to (\mathcal{X}_l, \delta_i)$ (using $\tau^\diamond_l$-iteration) that semiconjugates $\tau^\diamond_l$ to a ψ_l-equivariant λ-homothety f_l on $(\mathcal{X}_l, \delta_i)$; moreover, $(\mathcal{X}_l, \delta_i)$ is the associated $\mathcal{G}_l$-forest for the pseudometric δ_i on $\mathcal{Y}^*_l$. The following claim is a variation of Theorem II.9 that determines limits of iterated turns over $\mathcal{T}^\diamond_l$:

Claim IV.1. *Choose a nondegenerate component $T^\diamond \subset \mathcal{T}^\diamond_l$, corresponding components $G \subset \mathcal{G}_l$, $X \subset \mathcal{X}_l$, and a positive iterate ψ_l^k that preserves G. Let $\tilde{f} \colon (X, \delta_i) \to (X, \delta_i)$ be the φ-equivariant λ^k-homothety, where φ is in the outer class $[\psi_l^k\big|_F]$. Finally, for $\nu = 1, 2$, pick $p_\nu \in T^\diamond$ and $x_\nu \in F$.*

The point $p_{\nu,m} \coloneqq \varphi^{-1}(x_\nu) \cdots \varphi^{-m}(x_\nu) \cdot p_\nu$ in $(T^\diamond \varphi^m, \lambda^{-mk} \delta^\diamond_i)$ converges to $\star_\nu$ in $(\overline{X}, \delta_i)$ as $m \to \infty$, where $\star_\nu$ is the unique fixed point of $x_\nu^{-1} \cdot \tilde{f}$ in the metric completion $(\overline{X}, \delta_i)$.

Sketch of proof. Let $\pi^\diamond_l \colon (T^\diamond, \delta^\diamond_i) \to (X, \delta_i)$ be the constructed equivariant metric map and $\tilde{\tau}^\diamond_l \colon (T^\diamond, \delta^\diamond_i) \to (T^\diamond, \delta^\diamond_i)$ the φ-equivariant λ^k-Lipschitz "PL-map" induced by the simplicial automorphisms $(\tau_{l,j})_{j=1}^{n_l-1}$ and σ_l and ψ_{l+1}-equivariant λ-homothety f_{l+1}. The proof is the same as in Proposition II.4 using $\pi^\diamond_l$, $\tilde{\tau}^\diamond_l$, and the φ-equivariant λ^k-homothety $\tilde{f}$. □

By induction, $(\mathcal{X}, \delta_i) \coloneqq (\mathcal{X}_1, \delta_i)$ is the associated $\mathcal{F}$-forest for the pseudometric δ_i on $\mathcal{Y}^*_1$. We can now state a variation of Lemma V.6 whose proof is sketched in Section V.3:

Lemma V.9. *Let $\psi \colon \mathcal{F} \to \mathcal{F}$ be an exponentially growing automorphism with a descending sequence $(\mathcal{Y}_i, \delta_i)_{i=1}^n$ of limit forests, $\mathcal{L}^+_{\mathcal{Z}_i}[\psi_i] \subset \mathbb{R}(\mathcal{G}_i, \mathcal{Z}_i)$ the stable laminations for $(\mathcal{Y}_i, \delta_i)$, $\mathcal{H}_i[\psi]$ the support of the closure $\mathcal{L}^+_i[\psi]$ of $\mathcal{L}^+_{\mathcal{Z}_i}[\psi_i]$ in $\mathbb{R}(\mathcal{F})$, $(\mathcal{Y}^*, \oplus_{j=1}^k \delta_{\iota(j)})$ the topmost limit forest for $(\mathcal{Y}_i, \delta_i)_{i=1}^n$, and $(\mathcal{X}', \delta')$ a minimal $\mathcal{F}$-forest with trivial arc stabilizers. Pick some $i \in \{\iota(1), \ldots, \iota(k)\}$ and let $(\mathcal{X}, \delta_i)$ be the associated $\mathcal{F}$-forest for the pseudometric δ_i on $\mathcal{Y}^*$.*

If $\mathcal{Z}_i$, $\mathcal{H}_l[\psi]$ $(l < i)$ are $\mathcal{X}'$-elliptic and $(\mathcal{X}', \delta')$ is equivariantly isometric to $(\mathcal{X}'\psi, s^{-1}\delta')$ for some $s > 1$, then $(\mathcal{X}', \delta')$ is equivariantly isometric to $(\mathcal{X}, c\,\delta_i)$ for some $c > 0$.

IV.2 Universality of topmost expanding forest

Finally, we prove topmost limit forests are universal:

Theorem IV.2. *Let $\psi\colon \mathcal{F} \to \mathcal{F}$ be an exponentially growing automorphism. Any topmost $[\psi]$-expanding forests has a unique equivariant dilation to the topmost limit forest for $[\psi]$.*

Proof. Let $(\mathcal{Y}_i, \delta_i)_{i=1}^n$ be a descending sequence of limit forests for $[\psi]$, $\mathcal{L}^+_{\mathcal{Z}_i}[\psi_i] \subset \mathbb{R}(\mathcal{G}_i, \mathcal{Z}_i)$ the stable laminations for $(\mathcal{Y}_i, \delta_i)$, $\mathcal{H}_i[\psi]$ the support of the closure $\mathcal{L}^+_i[\psi]$ of $\mathcal{L}^+_{\mathcal{Z}_i}[\psi_i]$ in $\mathbb{R}(\mathcal{F})$, $(\mathcal{Y}^*, \oplus_{j=1}^k \delta_{\iota(j)})$ the topmost limit forest for $(\mathcal{Y}_i, \delta_i)_{i=1}^n$, and $(\mathcal{Y}', \oplus_{j=1}^k \delta'_j)$ a topmost $[\psi]$-expanding forest. Pick some $j \in \{1, \dots k\}$ and let $(\mathcal{X}, \delta_{\iota(j)})$ (resp. $(\mathcal{X}', \delta'_j)$) be the associated $\mathcal{F}$-forest for the pseudometric $\delta_{\iota(j)}$ on $\mathcal{Y}^*$ (resp. δ'_j on $\mathcal{Y}'$). By Theorem III.7(2), the $\mathcal{F}$-forest $(\mathcal{X}', \delta'_j)$ is equivariantly isometric to $(\mathcal{X}'\psi, s^{-1}\delta'_j)$ for some $s > 1$. Note that $\mathcal{G}[\mathcal{X}] = \mathcal{G}[\mathcal{X}']$, i.e. the two $\mathcal{F}$-forests have the same elliptic elements, by Theorem III.7(3). In particular, $\mathcal{Z}_{\iota(j)}$ and $\mathcal{H}_l[\psi]$ $(l < \iota(j))$ are $\mathcal{X}'$-elliptic. By Lemma V.9, $(\mathcal{X}', \delta'_j)$ is equivariantly isometric to $(\mathcal{X}, c\,\delta_{\iota(j)})$ for some $c > 0$. As the index j was arbitrary, the uniqueness of the equivariant metric blow-up construction implies $(\mathcal{Y}', \oplus_{j=1}^k \delta'_j)$ is uniquely equivariantly isometric to $\left(\mathcal{Y}^*, \oplus_{j=1}^k c_j\, \delta'_{\iota(j)}\right)$ for some $c_j > 0$. □

V Convergence criteria

This chapter adapts then extends Section 7 of Levitt–Lustig's paper [LL03]; they gave complete details for the proof sketched by Bestvina–Feighn–Handel in [BFH97, Lemma 3.4].

V.1 Proof of Lemma V.3

Fix an automorphism $\psi\colon \mathcal{F} \to \mathcal{F}$ with an expanding irreducible train track $\tau\colon \mathcal{T} \to \mathcal{T}$. Let $\lambda \coloneqq \lambda[\tau]$, $(\mathcal{Y}_\tau, d_\infty)$ be the limit forest for $[\tau]$, $\pi\colon (\mathcal{T}, d_\tau) \to (\mathcal{Y}_\tau, d_\infty)$ the constructed equivariant metric PL-map, $\mathcal{L}^+[\tau] \subset \mathbb{R}(\mathcal{T})$ the stable lamination for $[\tau]$, and $k \geq 1$ the number of components of $\mathcal{L}^+[\tau]$. Suppose $f\colon (\mathcal{T}, d_\tau) \to (\mathcal{Y}, \delta)$ is an equivariant PL-map and $\mathcal{L}^+[\tau]$ is in the canonically embedded subspace $\mathbb{R}(\mathcal{Y}, \delta) \subset \mathbb{R}(\mathcal{T})$.

Claim V.1 (cf. [LL03, Lemma 7.1])**.** *There is a sequence $c(f)$ of positive constants c_i indexed by the components $\Lambda_i^+ \subset \mathcal{L}^+[\tau]$ such that*

$$\lim_{m\to\infty} \lambda^{-mk}\delta(f(\tau^{mk}(p)), f(\tau^{mk}(q))) = c_i\, d_\infty(\pi(p), \pi(q))$$

for any leaf segment $[p, q]$ of Λ_i^+.

Any two equivariant PL-maps $f, g\colon (\mathcal{T}, d_\tau) \to (\mathcal{Y}, \delta)$ are a bounded δ-distance apart and $c(f) = c(g)$. So we can define $c(\mathcal{Y}, \delta) \coloneqq c(f)$; note that $c(\mathcal{Y}, s\,\delta) = s\, c(\mathcal{Y}, \delta)$ for $s > 0$. Without loss of generality, rescale the metric δ so that f is an equivariant metric PL-map.

Proof. Let $\nu^R \coloneqq \nu^R[\tau]$ (resp. $\nu^L \coloneqq \nu^L[\tau]$) be the unique positive right (resp. left) eigenvector for the irreducible transition matrix $A \coloneqq A[\tau]$ whose sum of entries is 1. Suppose $[p, q]$ is a leaf segment (of a component $\Lambda_i^+ \subset \mathcal{L}^+[\tau]$) with endpoints at vertices of $\mathcal{T}$ and let $v \coloneqq v[p, q]$ be the vector counting the occurences of $[e]$ in $[p, q]$: $[e]$ is an $\mathcal{F}$-orbit of edges in $\mathcal{T}$; the entries of $v = (v_e)$ are indexed by the $\mathcal{F}$-orbits $[e]$; and v_e is the number of translates of e in $[p, q]$. The train track property gives us $v^{(m)} \coloneqq v[\tau^m(p), \tau^m(q)] = A^m v$. Then, as $[p, q]$ is a leaf segment, the positive entries of $v^{(mk)}$ are indexed in the same block $\mathcal{B}_i = \mathcal{B}(\Lambda_i^+)$ for all $m \geq 0$. By Perron's theorem, if $[e]$ is in the block $\mathcal{B}_i$, then

$$\lim_{m\to\infty} \frac{v_e^{(mk)}}{\lambda^{mk}\langle \nu^L, v\rangle_i} = \frac{\nu_e^R}{\langle \nu^L, \nu^R\rangle_i}, \text{ where } \langle\cdot,\cdot\rangle_i \text{ is the dot product in the block } \mathcal{B}_i.$$

For small $\epsilon > 0$, fix $m_\epsilon \gg 1$ such that $\delta_e(m_\epsilon) \coloneqq \delta(f(\tau^{m_\epsilon k}(p_e)), f(\tau^{m_\epsilon k}(q_e))) > \epsilon^{-1}C[f]$ for every edge $e = [p_e, q_e]$ in $\mathcal{T}$ — we need the assumption $\mathcal{L}^+[\tau] \subset \mathbb{R}(\mathcal{Y}, \delta)$ for this. The interval $[\tau^{(m_\epsilon+m)k}(p), \tau^{(m_\epsilon+m)k}(q)]$ is a union of $v_e^{(mk)}$-many translates of $\tau^{m_\epsilon k}(e)$, as $[e]$ ranges over all the orbits of edges in $\mathcal{T}$. In $\mathcal{Y}$, we get

$$\sum_{[e]\subset\mathcal{T}} v_e^{(mk)}(\delta_e(m_\epsilon) - 2C[f]) \leq \delta(f(\tau^{(m_\epsilon+m)k}(p)), f(\tau^{(m_\epsilon+m)k}(q))) \leq \sum_{[e]\subset\mathcal{T}} v_e^{(mk)}\delta_e(m_\epsilon).$$

Divide by $\lambda^{(m_\epsilon+m)k} d_\infty(\pi(p),\pi(q)) = \lambda^{(m_\epsilon+m)k}\langle \nu^L, v\rangle_i$ and let $m \to \infty$:

$$\frac{1-2\epsilon}{\langle \nu^L,\nu^R\rangle_i} \sum_{[e]\in\mathcal{B}_i} \nu_e^R \frac{\delta_e(m_\epsilon)}{\lambda^{m_\epsilon k}} \le \liminf_{m\to\infty} \frac{\delta(f(\tau^{mk}(p)), f(\tau^{mk}(q)))}{\lambda^{mk} d_\infty(\pi(p),\pi(q))}$$
$$\le \limsup_{m\to\infty} \frac{\delta(f(\tau^{mk}(p)), f(\tau^{mk}(q)))}{\lambda^{mk} d_\infty(\pi(p),\pi(q))} \le \frac{1}{\langle \nu^L,\nu^R\rangle_i} \sum_{[e]\in\mathcal{B}_i} \nu_e^R \frac{\delta_e(m_\epsilon)}{\lambda^{m_\epsilon k}}$$

Since f is a metric map, we have $\lambda^{-m_\epsilon k}\delta_e(m_\epsilon) \le \nu_e^L$. So the lim inf and lim sup above are real, equal, and depends only on the block $\mathcal{B}_i$ for Λ_i^+. If ϵ is small, then $\epsilon^{-1}C[f] > 2C[f]+L$ for some $L>0$; by bounded cancellation,

$$c_i := \lim_{m\to\infty} \frac{\delta(f(\tau^{mk}(p)), f(\tau^{mk}(q)))}{\lambda^{mk} d_\infty(\pi(p),\pi(q))} \ge \lim_{m\to\infty} \frac{\|v^{(mk)}\|_1 L}{\lambda^{(m_\epsilon+m)k}\langle \nu^L, v\rangle_i} \ge \frac{\nu_e^R L}{\lambda^{m_\epsilon k}\langle \nu^L,\nu^R\rangle_i} > 0,$$

where $\|v^{(m)}\|_1$ is the sum of the entries in $v^{(m)}$ and $[e]$ is in the same block as $[p,q]$.

We now relax the restriction that $[p,q]$ is an edge-path, i.e. p,q need not be vertices. For $m\ge 0$, let $[\bar p_m, \bar q_m]$ be the shortest edge-path containing $[\tau^{mk}(p), \tau^{mk}(q)]$; for $m,m'\ge 0$,

$$\frac{\delta(f(\tau^{mk}(\bar p_{m'})), f(\tau^{mk}(\bar q_{m'}))) - \lambda^{mk}2}{\lambda^{mk}(d_\infty(\pi(\bar p_{m'}),\pi(\bar q_{m'}))+2)} \le \frac{\delta(f(\tau^{(m+m')k}(p)), f(\tau^{(m+m')k}(q)))}{\lambda^{(m+m')k} d_\infty(\pi(p),\pi(q))}$$
$$\le \frac{\delta(f(\tau^{mk}(\bar p_{m'})), f(\tau^{mk}(\bar q_{m'}))) + \lambda^{mk}2}{\lambda^{mk}(d_\infty(\pi(\bar p_{m'}),\pi(\bar q_{m'}))-2)}.$$

Both upper and lower bounds converge to c as $m', m\to\infty$: $[\bar p_{m'}, \bar q_{m'}]$ is a leaf segment with endpoints at vertices of $\mathcal{T}$, so

$$\lim_{m'\to\infty}\lim_{m\to\infty} \frac{\delta(f(\tau^{mk}(\bar p_{m'})), f(\tau^{mk}(\bar q_{m'}))) \mp \lambda^{mk}2}{\lambda^{mk}(d_\infty(\pi(\bar p_{m'}),\pi(\bar q_{m'}))\pm 2)} = \lim_{m'\to\infty} \frac{c_i\, d_\infty(\pi(\bar p_{m'}),\pi(\bar q_{m'})) \mp 2}{d_\infty(\pi(\bar p_{m'}),\pi(\bar q_{m'}))\pm 2} = c_i.$$

□

The next step is extending the claim to all intervals $[p,q]\subset\mathcal{T}$. Set $(c_i)_{i=1}^k := c(\mathcal{Y},\delta)$ and let $d_\infty = \oplus_{i=1}^k d_\infty^{(i)}$ be the factorization indexed by the components $\Lambda_i^+ \subset \mathcal{L}^+[\tau]$. For convenience, replace ψ with its iterate ψ^k, τ with τ^k, and λ with λ^k.

Claim V.2 (cf. [LL03, Lemma 7.2])**.** *For any $p_1, p_2 \in \mathcal{T}$,*

$$\lim_{m\to\infty} \lambda^{-m}\delta(f(\tau^m(p_1)), f(\tau^m(p_2))) = \sum_{i=1}^k c_i\, d_\infty^{(i)}(\pi(p_1),\pi(p_2)).$$

Proof. Let $[p_1,p_2]$ be an interval in $\mathcal{T}$ and $N(p_1,p_2)$ the number of vertices in (p_1,p_2). If $\tau^m(p_1) = \tau^m(p_2)$ for some $m\ge 0$, then the claim holds automatically. So we may assume

$\tau^m(p_1) \neq \tau^m(p_2)$ for $m \geq 0$. For a given $m' \geq 0$, let $[\tau^{m'}(p_1), \tau^{m'}(p_2)]$ be a concatenation of $N'+1$ leaf segments $[q_j, q_{j+1}]_{j=0}^{N'}$ (of $\Lambda^+_{i(j)} \subset \mathcal{L}^+[\tau]$) for some $N' \leq N(p_1,p_2)$ and $i(j) \in \{1,\ldots,k\}$, where $q_0 = \tau^{m'}(p_1)$ and $q_{N'+1} = \tau^{m'}(p_2)$.

Since f is a metric map with cancellation constant $C[f]$, we get

$$
\begin{aligned}
0 &\leq \left(\sum_{j=0}^{N'} \delta(f(\tau^m(q_j)), f(\tau^m(q_{j+1})))\right) - \delta(f(\tau^{m+m'}(p_1)), f(\tau^{m+m'}(p_2))) \\
&\leq \left(\sum_{j=0}^{N'} d_\tau(\tau^m(q_j), \tau^m(q_{j+1}))\right) - d_\tau(\tau^{m+m'}(p_1), \tau^{m+m'}(p_2)) + 2N'\, C[f].
\end{aligned}
$$

Note that $\sum_{j=0}^{N'} d_\tau(\tau^m(q_j), \tau^m(q_{j+1})) \leq \lambda^m d_\tau(\tau^{m'}(p_1), \tau^{m'}(p_2))$ as τ is λ-Lipschitz on $(\mathcal{T}, d_\tau)$. Multiply everything by $\lambda^{-(m+m')}$ then apply Claim V.1 and definition of d_∞:

$$
\begin{aligned}
0 &\leq \left(\lambda^{-m'} \sum_{j=0}^{N'} c_{i(j)}\, d_\infty(\pi(q_j), \pi(q_{j+1}))\right) - \liminf_{m\to\infty} \lambda^{-(m+m')}\delta(f(\tau^{m+m'}(p_1)), f(\tau^{m+m'}(p_2))) \\
&\leq \lambda^{-m'} d_\tau(\tau^{m'}(p_1), \tau^{m'}(p_2)) - d_\infty(\pi(p_1), \pi(p_2)).
\end{aligned}
$$

But $d_\infty(\pi(q_j), \pi(q_{j+1})) = d_\tau(q_j, q_{j+1})$ since $[q_i, q_{i+1}]$ is a leaf segment; so the sum in the inequality is $\sum_{i=1}^k c_i\, d_\tau^{(i)}(\tau^{m'}(p_1), \tau^{m'}(p_2))$. We can now prove the claim:

$$
\begin{aligned}
&\limsup_{m\to\infty} \left| \lambda^{-(m+m')}\delta(f(\tau^{m+m'}(p_1)), f(\tau^{m+m'}(p_2))) - \sum_{i=1}^k c_i\, d_\infty^{(i)}(\pi(p_1), \pi(p_2)) \right| \\
&\leq \lambda^{-m'} \sum_{i=1}^k c_i\, d_\tau^{(i)}(\tau^{m'}(p_1), \tau^{m'}(p_2)) - \liminf_{m\to\infty} \lambda^{-(m+m')}\delta(f(\tau^{m+m'}(p_1)), f(\tau^{m+m'}(p_2))) \\
&\qquad + \lambda^{-m'} \sum_{i=1}^k c_i\, d_\tau^{(i)}(\tau^{m'}(p_1), \tau^{m'}(p_2)) - \sum_{i=1}^k c_i\, d_\infty^{(i)}(\pi(p_1), \pi(p_2)) \\
&\leq \sum_{i=1}^k (1+c_i)\left(\lambda^{-m'} d_\tau^{(i)}(\tau^{m'}(p_1), \tau^{m'}(p_2)) - d_\infty^{(i)}(\pi(p_1), \pi(p_2))\right);
\end{aligned}
$$

the upper bound vanishes as $m' \to \infty$ by the definition of $d_\infty^{(i)}$:

$$
\lim_{m'\to\infty} \limsup_{m\to\infty} \left| \lambda^{-(m+m')}\delta(f(\tau^{m+m'}(p_1)), f(\tau^{m+m'}(p_2))) - \sum_{i=1}^k c_i\, d_\infty^{(i)}(\pi(p_1), \pi(p_2)) \right| = 0. \quad \square
$$

Like in our construction of limit forests (Section II.1), let δ_m^* be the pullback of $\lambda^{-m}\delta$ via $f \circ \tau^m$ for $m \geq 0$. Then δ_m^* is an $\mathcal{F}$-invariant pseudometric on $\mathcal{T}$ whose quotient metric space is equivariantly isometric to $(\mathcal{Y}\psi^m, \lambda^{-m}\delta)$. By Claim V.2, the (pointwise) limit $\lim_{m\to\infty} \delta_m^*$ is the pullback of $\oplus_{i=1}^k c_i\, d_\infty^{(i)}$ via π. In other words, the sequence $(\mathcal{Y}\psi^m, \lambda^{-m}\delta)_{m\geq 0}$ converges to $(\mathcal{Y}_\tau, \oplus_{i=1}^k c_i\, d_\infty^{(i)})$ and we are done:

Lemma V.3 (cf. [BFH97, Lemma 3.4])**.** *Let $\psi\colon \mathcal{F} \to \mathcal{F}$ be an automorphism, $\tau\colon \mathcal{T} \to \mathcal{T}$ an expanding irreducible train track for ψ, $(\mathcal{Y}_\tau, d_\infty)$ the limit forest for $[\tau]$, and $\lambda \coloneqq \lambda[\tau]$.*

If $(\mathcal{T}, d_\tau) \to (\mathcal{Y}, \delta)$ is an equivariant PL-map and the k-component lamination $\mathcal{L}^+[\tau]$ is in $\mathbb{R}(\mathcal{Y}, \delta) \subset \mathbb{R}(\mathcal{T})$, then the sequence $(\mathcal{Y}\psi^{mk}, \lambda^{-mk}\delta)_{m\geq 0}$ converges to $(\mathcal{Y}_\tau, \oplus_{i=1}^k c_i\, d_\infty^{(i)})$, where $d_\infty = \oplus_{i=1}^k d_\infty^{(i)}$ and $c_i > 0$. □

V.2 Proof of Lemma V.6

Fix an exponentially growing automorphism $\psi\colon \mathcal{F} \to \mathcal{F}$. Let $(\tau_i\colon \mathcal{T}_i \to \mathcal{T}_i)_{i=1}^n$ be a descending sequence of irreducible train tracks for $[\psi]$ rel. $\mathcal{Z}$, $\lambda \coloneqq \lambda[\tau_n]$, $\mathcal{T}^\circ$ be an equivariant blow-up of the free splittings $(\mathcal{T}_i)_{i=1}^n$, $\mathcal{L}_{\mathcal{Z}}^+[\psi] \subset \mathbb{R}(\mathcal{F}, \mathcal{Z})$ the k-component stable laminations for $[\psi]$ rel. $\mathcal{Z}$, $(\mathcal{Y}, \delta)$ the limit forest for $[\tau_i]_{i=1}^n$, $\pi^\circ\colon (\mathcal{T}^\circ, d^\circ) \to (\mathcal{Y}, \delta)$ the equivariant metric map constructed using τ°-iteration, and $\delta = \oplus_{j=1}^k \delta_j$ the factorization indexed by components $\Lambda_j^+ \subset \mathcal{L}_{\mathcal{Z}}^+[\psi]$. For convenience, replace ψ with ψ^k, τ° with $\tau^{\circ k}$, and λ with λ^k.

Suppose $(\mathcal{Y}', \delta')$ is a minimal $\mathcal{F}$-forest with trivial arc stabilizers, $\mathcal{Z}$ is $\mathcal{Y}'$-elliptic, and $\mathcal{L}_{\mathcal{Z}}^+[\psi]$ is in $\mathbb{R}(\mathcal{Y}', \delta') \subset \mathbb{R}(\mathcal{F}, \mathcal{Z})$. Let $(\mathcal{T}_i^\circ, d^\circ)$ be the characteristic subforest of $(\mathcal{T}^\circ, d^\circ)$ for $\mathcal{F}_i$, $(\mathcal{Y}_i', \delta')$ the characteristic subforest of $(\mathcal{Y}', \delta')$ for $\mathcal{F}_i$, and $f_i\colon (\mathcal{T}_i^\circ, d^\circ) \to (\mathcal{Y}_i', \delta')$ an equivariant metric PL-map. By applying Claim V.1 to f_n, we can set $(c_j)_{j=1}^k \coloneqq c(f_n) > 0$.

Claim V.4. *For $1 \leq i \leq n$, $k' \geq 1$, and any iterated turn $[p_{1,m}, p_{2,m}]_{m\geq 0}$ over $\mathcal{T}_i^\circ$ rel. $\psi_i^{k'}$,*

$$\lim_{m\to\infty} \lambda^{-mk'} \delta'(f_i(p_{1,m}), f_i(p_{2,m})) = \sum_{j=1}^k c_j\, \delta_j(\star_1, \star_2),$$

where $[\star_1, \star_2] \subset \overline{\mathcal{Y}}_i$ is the limit of the iterated turn (Theorem II.9).

We will prove that the following Claim V.5 at level $i \leq n$ implies Claim V.4 at level i.

Claim V.5. *For $1 \leq i \leq n$ and any $p_1, p_2 \in \mathcal{T}_i^\circ$,*

$$\lim_{m\to\infty} \lambda^{-m} \delta'(f_i(\tau_i^{\circ m}(p_1)), f_i(\tau_i^{\circ m}(p_2))) = \sum_{j=1}^k c_j\, \delta_j(\pi^\circ(p_1), \pi^\circ(p_2)).$$

In the base case, Claim V.5 at level n is Claim V.2.

Proof of Claim V.4 at level $i \leq n$, assuming Claim V.5 at level i.

Recall that $[\psi_i]$ is the restriction of $[\psi]$ to $\mathcal{F}_i$. Let $[p_{1,m}, p_{2,m}]_{m\geq 0}$ be an iterated turn over $\mathcal{T}_i^\circ$ rel. $\psi_i^{k'}$. For $m', m \geq 0$, the term $[p_{1,m+m'}, p_{2,m+m'}]$ is covered by $(2m+1)$ intervals:

$$
\begin{aligned}
[q_{m-j}, q_{m-j+1}] &\coloneqq [\tau_i^{\circ(m-j)k'}(p_{1,m'+j}), \tau_i^{\circ(m-j+1)k'}(p_{1,m'+j-1})] \quad \text{for } 1 \leq j \leq m,\\
[q_m, q_{m+1}] &\coloneqq [\tau_i^{\circ mk'}(p_{1,m'}), \tau_i^{\circ mk'}(p_{2,m'})], \text{ and}\\
[q_{m+j}, q_{m+j+1}] &\coloneqq [\tau_i^{\circ(m-j+1)k'}(p_{2,m'+j-1}), \tau_i^{\circ(m-j)k'}(p_{2,m'+j})] \quad \text{for } 1 \leq j \leq m.
\end{aligned}
$$

Since f_i is a metric map with cancellation constant $C[f_i]$,

$$
\begin{aligned}
&\left|\delta'(f_i(p_{1,m+m'}), f_i(p_{2,m+m'})) - \delta'(f_i(\tau_i^{\circ mk'}(p_{1,m'})), f_i(\tau_i^{\circ mk'}(p_{2,m'})))\right|\\
&\leq \left(\sum_{j=0}^{2m} \delta'(f_i(q_j), f_i(q_{j+1}))\right) - \delta'(f_i(q_0), f_i(q_{2m+1})) + \sum_{\substack{j=0\\ j\neq m}}^{2m} \delta(f_i(q_j), f_i(q_{j+1}))\\
&\leq d^\circ(q_m, q_{m+1}) - d^\circ(q_0, q_{2m+1}) + 4m\, C[f_i] + 2\sum_{\substack{j=0\\ j\neq m}}^{2m} d^\circ(q_j, q_{j+1}).
\end{aligned}
$$

As in the proof of Proposition II.4, the interval $[\tau_i^{\circ(m-j)k'}(p_{\iota,m'+j}), \tau_i^{\circ(m-j+1)k'}(p_{\iota,m'+j-1})]$ is a translate of $[\tau_i^{\circ(m-j)k'}(\psi_i^{m'k'}(x_i)\cdot p_{\iota,m'}), \tau_i^{\circ(m-j+1)k'}(p_{\iota,m'})]$ for $1 \leq j \leq m$ and $\iota = 1, 2$. Set $D \coloneqq \max\{d^\circ(\psi_i^{m'k'}(x_\iota)\cdot p_{\iota,m'}, \tau_i^{\circ k'}(p_{\iota,m'})) : \iota = 1, 2\}$ and $D' \coloneqq \frac{D}{\lambda^{k'}-1}$. Since τ_i° is λ-Lipschitz with respect to d°, the first term $d^\circ(q_m, q_{m+1})$ is at most $\lambda^{mk'} d^\circ(p_{1,m'}, p_{2,m'})$ and the last sum at most $\lambda^{mk} 2D'$. Multiply everything by $\lambda^{-(m+m')k'}$ then apply Claim V.5 at level i and Theorem II.9 — the needed limit is only explicitly stated in Proposition II.4:

$$
\begin{aligned}
&\limsup_{m\to\infty} \lambda^{-m'k'} \left|\lambda^{-mk'}\delta'(f_i(p_{1,m+m'}), f(p_{2,m+m'})) - \textstyle\sum_{j=1}^k c_j\, \delta_j(\pi^\circ(p_{1,m'}), \pi^\circ(p_{2,m'}))\right|\\
&\qquad \leq \lambda^{-m'k'} d^\circ(p_{1,m'}, p_{2,m'}) - \delta(\star_1, \star_2) + \lambda^{-m'k'} 4D'.
\end{aligned}
$$

We can now conclude the proof:

$$
\begin{aligned}
&\limsup_{m\to\infty} \left|\lambda^{-(m+m')k'}\delta'(f_i(p_{1,m+m'}), f_i(p_{2,m+m'})) - \textstyle\sum_{j=1}^k c_j\, \delta_j(\star_1, \star_2)\right|\\
&\leq \limsup_{m\to\infty} \lambda^{-m'k'} \left|\lambda^{-mk'}\delta'(f_i(p_{1,m+m'}), f_i(p_{2,m+m'})) - \textstyle\sum_{j=1}^k c_j\, \delta_j(\pi^\circ(p_{1,m'}), \pi^\circ(p_{2,m'}))\right|\\
&\qquad + \sum_{j=1}^k c_j \left|\lambda^{-m'k'}\delta_j(\pi^\circ(p_{1,m'}), \pi^\circ(p_{2,m'})) - \delta_j(\star_1, \star_2)\right|\\
&\leq \lambda^{-m'k} d^\circ(p_{1,m'}, p_{2,m'}) - \delta(\star_1, \star_2) + \lambda^{-m'k} 4D'\\
&\qquad + \sum_{j=1}^k c_j \left|\lambda^{-m'k}\delta_j(\pi^\circ(p_{1,m'}), \pi^\circ(p_{2,m'})) - \delta_j(\star_1, \star_2)\right|;
\end{aligned}
$$

the upper bound vanishes as $m' \to \infty$ by Theorem II.9 again. $\square$

We now prove that Claim V.5 at level $i+1 \le n$ implies Claim V.5 at level i. In light of the previous proof, we are free to also invoke Claim V.4 at level $i+1$ in this case.

Proof of Claim V.5 at level $i < n$, assuming Claim V.5 at level $i+1$.

Adapting the proof of Theorem II.10, suppose $[\tau_i^{k'}]$ (for some $k' \ge 1$) fixes all $\mathcal{F}_i$-orbits of vertices and edges in $\mathcal{T}_i$. Let $[p_1, p_2] \subset \mathcal{T}_i^\circ$ be an edge-path whose endpoints are in $[\tau_i^\circ]$-periodic $\mathcal{F}_i$-orbits $[p_1], [p_2]$. Since this claim holds at level $i+1$ and f_i, f_{i+1} are a bounded δ'-distance apart on $\mathcal{T}_i^\circ$, we may assume $[p_1, p_2]$ contains an i^{th} stratum edge. So $[p_1, p_2]$ projects to an edge-path $e_1 \cdots e_N$ in $\mathcal{T}_i$ for some $N \ge 1$. For $1 \le j < N$ and $m \ge 0$, let $w_{j,m}$ be the closed interval in $\mathcal{F}_i \cdot \mathcal{T}_{i+1}^\circ$ between $\tau_i^m(e_j)$ and $\tau_i^m(e_{j+1})$. For $0 \le r < k'$, the sequence $(w_{j,mk'+r})_{m\ge 0}$ is an iterated turn over $\mathcal{T}_{i+1}^\circ$ rel. $\psi_{i+1}^{k'}$. As the $\mathcal{F}_i$-orbits $[p_1], [p_2]$ are $[\tau_i^\circ]$-periodic, the initial and terminal segments of $[\tau_i^{\circ mk'+r}(p_1), \tau_i^{\circ mk'+r}(p_2)]$ in $\mathcal{F}_i \cdot \mathcal{T}_{i+1}^\circ$ form iterated turns over $\mathcal{T}_{i+1}^\circ$ rel. $\psi_{i+1}^{k'}$ denoted $(w_{0,mk'+r})_{m\ge 0}$ and $(w_{N,mk'+r})_{m\ge 0}$ respectively.

For $0 \le j \le N$ and $m \ge 0$, set $[q_{1,m}^j, q_{2,m}^j] := w_{j,m}$; so $[\tau_i^{\circ m}(p_1), \tau_i^{\circ m}(p_2)]$ is the concatenation of intervals $[q_{1,m}^j, q_{2,m}^j]_{j=0}^N$ and $[q_{2,m}^{j-1}, q_{1,m}^j]_{j=1}^N$. For $0 \le r < k'$, let $[\star_{1,r}^j, \star_{2,r}^j]$ be the limit interval of $[q_{1,mk'+r}^j, q_{2,mk'+r}^j]_{m\ge 0}$ in (a translate of) $\widehat{\mathcal{Y}}_{i+1} \subset \mathcal{Y}_i$ given by Theorem II.9. In particular, $\star_{2,r}^{j-1} = \star_{1,r}^j$ for $1 \le j \le N$; therefore, the interval $[\pi^\circ(\tau_i^{\circ r}(p_1)), \pi^\circ(\tau_i^{\circ r}(p_2))]$ is the concatenation of limit intervals $[\star_{1,r}^j, \star_{2,r}^j]_{j=0}^N$. Let h_i be the ψ-equivariant λ-homothety on $(\mathcal{Y}_i, \delta)$; then the semiconjugacy states that $\pi^\circ(\tau_i^{\circ r}(p_\iota)) = h_i^r(\pi^\circ(p_\iota))$ for $\iota = 1, 2$.

Since f_i is a metric map with cancellation constant $C[f_i]$,

$$
\begin{aligned}
&\left| \delta'(f_i(\tau_i^{\circ mk'+r}(p_1)), f(\tau_i^{\circ mk'+r}(p_2))) - \sum_{j=0}^{N} \delta'(f_i(q_{1,mk'+r}^j), f_i(q_{2,mk'+r}^j)) \right| \\
&\quad \le \left(\sum_{j=0}^{N} \delta'(f_i(q_{1,mk'+r}^j), f_i(q_{2,mk'+r}^j)) \right) + 2\left(\sum_{j=1}^{N} \delta'(f_i(q_{2,mk'+r}^{j-1}), f_i(q_{1,mk'+r}^j)) \right) \\
&\qquad\quad - \delta'(f_i(\tau_i^{\circ mk'+r}(p_1)), f(\tau_i^{\circ mk'+r}(p_2))) \\
&\quad \le \left(\sum_{j=0}^{N} d^\circ(q_{1,mk'+r}^j, q_{2,mk'+r}^j) \right) + 2\left(\sum_{j=1}^{N} d^\circ(q_{2,mk'+r}^{j-1}, q_{1,mk'+r}^j) \right) \\
&\qquad\qquad - d^\circ(\tau_i^{\circ mk'+r}(p_1), \tau_i^{\circ mk'+r}(p_2)) + 4N\,C[f_i] \\
&\quad = 4N\,C[f_i] + \sum_{j=1}^{N} d^\circ(q_{2,mk'+r}^{j-1}, q_{1,mk'+r}^j).
\end{aligned}
$$

Since d° assigned the same length L_i to all edges of $\mathcal{T}_i$, the last sum is NL_i. Multiply everything by $\lambda^{-(mk'+r)}$ then apply Claim V.4 at level $i+1$:

$$\lim_{m\to\infty}\left|\lambda^{-(mk'+r)}\delta'(f_i(\tau_i^{\circ mk'+r}(p_1)), f(\tau_i^{\circ mk+r}(p_2))) - \lambda^{-r}\sum_{j=0}^{N}\sum_{j'=1}^{k} c_{j'}\,\delta_{j'}(\star_{1,r}^{j},\star_{2,r}^{j})\right| = 0.$$

This is the needed limit as $\delta_{j'}(\pi^\circ(p_1),\pi^\circ(p_2)) = \lambda^{-r}\sum_{j=0}^{N}\delta_{j'}(\star_{1,r}^{j},\star_{2,r}^{j})$ and r was arbitrary.

Now let $[p_1,p_2]$ be an arbitrary interval. If some $[\tau_i^{\circ m}(p_1),\tau_i^{\circ m}(p_2)]$ does not intersect an i^{th} stratum edge, then the semiconjugacy $\pi^\circ\circ\tau_i^\circ = h_i\circ\pi^\circ$ and Claim V.5 at level $i+1$ imply the claim. Assume $[\tau_i^{\circ m}(p_1),\tau_i^{\circ m}(p_2)]$ intersects an i^{th} stratum edge for $m\geq 0$. Let $[\bar p_{1,m},\bar p_{2,m}]$ be an edge-path extension of $[\tau_i^{\circ m}(p_1),\tau_i^{\circ m}(p_2)]$ whose endpoints are in $[\tau_i^\circ]$-periodic orbits and each "end" is extended by at most $B=B(\mathcal{T}_i^\circ)$. Thus

$$\begin{aligned}&\left|\lambda^{-(m+m')}\delta'(f_i(\tau_i^{\circ m+m'}(p_1)), f_i(\tau_i^{\circ m+m'}(p_2))) - \sum_{j=1}^{k} c_j\,\delta_j(\pi^\circ(p_1),\pi^\circ(p_2))\right|\\ &\quad\leq\left|\lambda^{-(m+m')}\delta'(f_i(\tau_i^{\circ m}(\bar p_{1,m'})), f_i(\tau_i^{\circ m}(\bar p_{2,m'}))) - \sum_{j=1}^{k} c_j\,\delta_j(\pi^\circ(p_1),\pi^\circ(p_2))\right| + \lambda^{-m'}2B.\end{aligned}$$

Since $[\bar p_{1,m},\bar p_{2,m}]$ is an edge-path whose endpoints are in $[\tau_i]$-periodic orbits, the upper bound converges as $m\to\infty$ by the first part of this proof.

To conclude,

$$\begin{aligned}&\limsup_{m\to\infty}\left|\lambda^{-(m+m')}\delta'(f_i(\tau_i^{\circ m+m'}(p_1)), f_i(\tau_i^{\circ m+m'}(p_2))) - \sum_{j=1}^{k} c_j\,\delta_j(\pi^\circ(p_1),\pi^\circ(p_2))\right|\\ &\quad\leq\sum_{j=1}^{k} c_j\left|\lambda^{-m'}\delta_j(\pi^\circ(\bar p_{1,m'}),\pi^\circ(\bar p_{2,m'})) - \delta_j(\pi^\circ(p_1),\pi^\circ(p_2))\right| + \lambda^{-m'}2B\\ &\quad\leq\sum_{j=1}^{k} c_j\left|\lambda^{-m'}\delta_j(\pi^\circ(\tau_i^{\circ m'}(p_1)),\pi^\circ(\tau_i^{\circ m'}(p_2))) - \delta_j(\pi^\circ(p_1),\pi^\circ(p_2))\right| + \lambda^{-m'}4B\\ &\quad=\lambda^{-m'}4B;\end{aligned}$$

the upper bound vanishes as $m'\to\infty$. □

This inductively proves Claim V.5 at level 1. The rest of the argument is the same as in the previous section. Let δ_m^* be pullback of $\lambda^{-m}\delta'$ via $f_1\circ\tau^{\circ m}$ for $m\geq 0$. By Claim V.5, the limit $\lim\limits_{m\to\infty}\delta_m^*$ is the pullback of $\oplus_{j=1}^{k} c_j\,\delta_j$ via π° and we are done:

Lemma V.6. *Let $\psi\colon \mathcal{F} \to \mathcal{F}$ be an exponentially growing automorphism with a descending sequence $(\tau_i\colon \mathcal{T}_i \to \mathcal{T}_i)_{i=1}^n$ of irreducible train tracks rel. $\mathcal{Z}$, $(\mathcal{Y}, \delta)$ the limit forest for $[\tau_i]_{i=1}^n$, $\lambda := \lambda[\tau_n]$, and $(\mathcal{Y}', \delta')$ a minimal $\mathcal{F}$-forest with trivial arc stabilizers.*

If $\mathcal{Z}$ is $\mathcal{Y}'$-elliptic and the k-component lamination $\mathcal{L}^+_{\mathcal{Z}}[\psi]$ is in $\mathbb{R}(\mathcal{Y}', \delta') \subset \mathbb{R}(\mathcal{F}, \mathcal{Z})$, then the limit of $(\mathcal{Y}'\psi^{mk}, \lambda^{-mk}\delta')_{m\geq 0}$ is $(\mathcal{Y}, \oplus_{j=1}^k c_j\,\delta_j)$, where $\delta = \oplus_{j=1}^k \delta_j$ and $c_j > 0$. □

V.3 Sketch of Lemma V.9

Fix an exponentially growing automorphism $\psi\colon \mathcal{F} \to \mathcal{F}$. Let $(\mathcal{Y}_i, \delta_i)_{i=1}^n$ be a descending sequence of limit forests for $[\psi]$, $\mathcal{L}^+_{\mathcal{Z}_i}[\psi_i] \subset \mathbb{R}(\mathcal{G}_i, \mathcal{Z}_i)$ the stable laminations for $(\mathcal{Y}_i, \delta_i)$, $\mathcal{H}_i[\psi]$ the support of the closure $\mathcal{L}^+_i[\psi]$ of $\mathcal{L}^+_{\mathcal{Z}_i}[\psi_i]$ in $\mathbb{R}(\mathcal{F})$, and $(\mathcal{Y}^*, \oplus_{j=1}^k \delta_{\iota(j)})$ the topmost limit forest for $(\mathcal{Y}_i, \delta_i)_{i=1}^n$. Pick some $i \in \{\iota(j) : 1 \leq j \leq k\}$ and let $(\mathcal{X}, \delta_i)$ be the associated $\mathcal{F}$-forest for the pseudometric δ_i on $\mathcal{Y}^*$ and $\mathcal{X}_l \subset \mathcal{X}$ the characteristic convex subsets for $\mathcal{G}_l$ $(1 \leq l \leq i)$. Recall that $(\mathcal{X}_i, \delta_i)$ was identified with the limit forest $(\mathcal{Y}_i, \delta_i)$ for $[\psi_i]$ rel. $\mathcal{Z}_i$.

Suppose $(\mathcal{X}', \delta')$ is a minimal $\mathcal{F}$-forest with trivial arc stabilizers, $\mathcal{Z}_i$ and $\mathcal{H}_l[\psi]$ $(l < i)$ are $\mathcal{X}'$-elliptic, and $(\mathcal{X}', \delta')$ is equivariantly isometric to $(\mathcal{X}'\psi, s^{-1}\delta')$ for some $s > 1$. Let $\mathcal{X}'_l \subset \mathcal{X}'$ be the characteristic convex subsets for $\mathcal{G}_l$ $(1 \leq l \leq i)$. Then the characteristic subforest $(\mathcal{X}'_i, \delta')$ for $\mathcal{G}_i$ is an expanding forest for $[\psi_i]$ rel. $\mathcal{Z}_i$. By Corollary II.11, $(\mathcal{X}'_i, \delta')$ can be identified with $(\mathcal{X}_i, c\,\delta_i) = (\mathcal{Y}_i, c\,\delta_i)$ for some $c > 0$; moreover, $s = \lambda$ is the scaling factor for the ψ_i-equivariant expanding homothety h_i on $(\mathcal{Y}_i, \delta_i)$.

For $1 \leq l < i$, let $(\mathcal{T}^\diamond_l, c\,\delta^\diamond_i)$ be an equivariant metric blow-up of the free splittings $(\mathcal{T}_{l,j})_{j=1}^{n_l-1}$, the skeleton $\mathcal{S}_l$, and the $\mathcal{G}_{l+1}$-forest $(\mathcal{X}_{l+1}, c\,\delta_i)$. The simplicial automorphisms $(\tau_{l,j})_{j=1}^{n_l-1}$ and σ_l induce a ψ_l-equivariant λ-Lipschitz map $\tau^\diamond_l$ on $(\mathcal{T}^\diamond_l, c\,\delta^\diamond_i)$ that linearly extends the ψ_{l+1}-equivariant λ-homothety f_{l+1} on $(\mathcal{X}_{l+1}, c\,\delta_i)$. Let $\pi^\diamond_l\colon (\mathcal{T}^\diamond_l, c\,\delta^\diamond_i) \to (\mathcal{X}_l, c\,\delta_i)$ be the constructed equivariant metric map that semiconjugates $\tau^\diamond_l$ to f_l and extends the inclusion $\mathcal{X}_{l+1} \subset \mathcal{X}_l$.

Since $\mathcal{H}_l[\psi]$ is $\mathcal{X}'_l$-elliptic, there is an equivariant map $g_l\colon (\mathcal{T}^\diamond_l, c\,\delta^\diamond_i) \to (\mathcal{X}'_l, \delta')$ that linearly extends the identification $(\mathcal{X}_{l+1}, c\,\delta_i) = (\mathcal{X}'_{l+1}, \delta')$; this map g_l is Lipschitz. Pick any free splitting $\mathcal{T}$ of $\mathcal{G}_l$ with $\mathcal{F}[\mathcal{T}] = \mathcal{Z}_i$. Then any equivariant map $\mathcal{T} \to \mathcal{T}^\diamond_l$ is surjective (by minimality) and composes with g_l to give (up to an equivariant homotopy rel. the vertices) an equivariant PL-map with a cancellation constant. So g_l must have a cancellation constant $C[g_l] \geq 0$. Here is a variation of Claim V.4 whose proof is essentially the same:

Claim V.7. *For $k \geq 1$ and any iterated turn $[p_{1,m}, p_{2,m}]_{m\geq 0}$ over $\mathcal{T}^\diamond_l$ rel. ψ^k_l,*

$$\lim_{m\to\infty} \lambda^{-mk}\delta'(g_l(p_{1,m}), g_l(p_{2,m})) = c\,\delta_i(\star_1, \star_2),$$

where $[\star_1, \star_2] \subset \overline{\mathcal{X}}_l$ is the limit of the iterated turn (Claim IV.1). □

The proof of this claim expectedly relies on a variation of Claim V.5:

Claim V.8. *For any $p_1, p_2 \in \mathcal{T}_l^\diamond$,*

$$\lim_{m\to\infty} \lambda^{-m}\delta'(g_l(\tau_l^{\diamond m}(p_1)), g_l(\tau_l^{\diamond m}(p_2))) = c\,\delta_i(\pi_l^\diamond(p_1), \pi_l^\diamond(p_2)).$$

Sketch of proof. Recall that $\mathcal{T}_{l,j}$ ($1 \le j < n_l$) is a free splitting of free factor systems $\mathcal{F}_{l,j}$ of $\mathcal{G}_l$. The skeleton $\mathcal{S}_l$ is a minimal simplicial $\mathcal{F}_{l,n_l}$-forest but not necessarily a free splitting (nor even a *small* splitting). We inductively ascend on the free factor systems $\mathcal{F}_{l,j}$ like we did in the proof of Claim V.5. Let $\mathcal{T}_{l,j}^\diamond \subset \mathcal{T}_l^\diamond$ be the characteristic convex subsets for $\mathcal{F}_{l,j}$.

Let $[p_1, p_2] \subset \mathcal{T}_{l,j}^\diamond$ be an edge-path whose endpoints are in $[\tau_l^\diamond]$-periodic $\mathcal{F}_{l,j}$-orbits $[p_1], [p_2]$. Since $(\mathcal{X}_{l+1}, c\,\delta_i)$ is the characteristic subforest of $(\mathcal{T}_l^\diamond, c\,\delta_i^\diamond)$ for $\mathcal{G}_{l+1}$, g_l extends the identification $(\mathcal{X}_{l+1}, c\,\delta_i) = (\mathcal{X}'_{l+1}, \delta')$, $\pi_l^\diamond$ extends the inclusion $\mathcal{X}_{l+1} \subset \mathcal{X}_l$, and $\tau_l^\diamond$ is an equivariant extension of the λ-homothety f_{l+1}, the claim holds if $[p_1, p_2]$ is contained in $\mathcal{X}_{l+1}$. So we are free to invoke Claim V.7 for iterated turns over $\mathcal{X}_{l+1}$. We may assume $[p_1, p_2]$ contains an edge of $\mathcal{S}_l$ if $j = n_l$ or of $\mathcal{T}_{l,j}$ if $j < n_l$. The rest of the proof proceeds just as with Claim V.5: for induction, assume the claim (and hence Claim V.7) holds for $\mathcal{T}_{l,j}^\diamond$ for $1 < j \le n_l$, then use the same argument to prove the claim holds for $\mathcal{T}_{l,j-1}^\diamond$. □

Thus the $\mathcal{G}_l$-forest $(\mathcal{X}_l, c\,\delta_i)$ is the limit of $(\mathcal{X}'_l\psi_l^m, \lambda^{-m}\delta')_{m\ge0}$. Yet $(\mathcal{X}'_l, \delta')$ is equivariantly isometric to $(\mathcal{X}'_l\psi_l, \lambda^{-1}\delta')$. So $(\mathcal{X}'_l, \delta')$ is equivariantly isometric to $(\mathcal{X}_l, c\,\delta_i)$. This concludes the induction step. By induction, $(\mathcal{X}', \delta')$ is equivariantly isometric to $(\mathcal{X}, c\,\delta_i)$:

Lemma V.9. *Let $\psi\colon \mathcal{F} \to \mathcal{F}$ be an exponentially growing automorphism with a descending sequence $(\mathcal{Y}_i, \delta_i)_{i=1}^n$ of limit forests, $\mathcal{L}^+_{\mathcal{Z}_i}[\psi_i] \subset \mathbb{R}(\mathcal{G}_i, \mathcal{Z}_i)$ the stable laminations for $(\mathcal{Y}_i, \delta_i)$, $\mathcal{H}_i[\psi]$ the support of the closure $\mathcal{L}^+_i[\psi]$ of $\mathcal{L}^+_{\mathcal{Z}_i}[\psi_i]$ in $\mathbb{R}(\mathcal{F})$, $(\mathcal{Y}^*, \oplus_{j=1}^k \delta_{\iota(j)})$ the topmost limit forest for $(\mathcal{Y}_i, \delta_i)_{i=1}^n$, and $(\mathcal{X}', \delta')$ a minimal $\mathcal{F}$-forest with trivial arc stabilizers. Pick some $i \in \{\iota(1), \ldots, \iota(k)\}$ and let $(\mathcal{X}, \delta_i)$ be the associated $\mathcal{F}$-forest for the pseudometric δ_i on $\mathcal{Y}^*$.*

If $\mathcal{Z}_i$, $\mathcal{H}_l[\psi]$ ($l < i$) are $\mathcal{X}'$-elliptic and $(\mathcal{X}', \delta')$ is equivariantly isometric to $(\mathcal{X}'\psi, s^{-1}\delta')$ for some $s > 1$, then $(\mathcal{X}', \delta')$ is equivariantly isometric to $(\mathcal{X}, c\,\delta_i)$ for some $c > 0$. □

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