Research Statement

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My area of research is *Geometric Group Theory* (GGT) — this field studies groups as geometric objects and it lies in the overlaps of group theory, topology, geometry, dynamics, and more... GGT was especially useful for low-dimensional topology following William Thurston's foundational work on surfaces and 3-manifolds; one of Thurston's results gave a canonical geometric decomposition of topological 3-manifolds that fiber over the circle in terms of the dynamics of a surface homeomorphism.

In many ways, free groups behave like surface groups and the tools Thurston developed for the mapping class group $\mathrm{MCG}(S)$ of a surface have inspired analogous tools for the outer automorphism group $\mathrm{Out}(F)$ of a free group. For example, surface homeomorphisms are related to 3-manifolds that fiber over the circle just as free group automorphisms $\phi : F \to F$ are related to *free-by-cyclic* groups $F \rtimes_\phi \mathbb{Z}$; Bestvina–Handel defined *irreducible* outer automorphisms of free groups as analogues for *pseudo-Anosov* mapping classes and used *train tracks* to study their dynamics just as Thurston did for pseudo-Anosovs; Culler–Vogtmann's outer space $CV(F)$, the space of marked metric graphs, is analogous to Teichmüller space $\mathcal{T}(S)$, the space of marked hyperbolic surfaces, and these spaces serve as models for $\mathrm{Out}(F)$ and $\mathrm{MCG}(S)$ respectively.

To add to this dictionary, I am looking for a free-by-cyclic analogue of Thurston's geometric decomposition in terms of the dynamics of a related free group automorphism. Along these lines, I have proven:

- (Theorem 2.1) All automorphisms $\phi\colon F \to F$ have a corresponding canonical decomposition of the free group F that is determined solely by the dynamics of ϕ.
- (Theorem 1.1) The property of an automorphism $\phi\colon F \to F$ being irreducible and atoroidal is a commensurability invariant of the free-by-cyclic group $F \rtimes_\phi \mathbb{Z}$.

The first statement is rather vague and more details will be given later. The second answers a question of Dowdall–Kapovich–Leininger on whether irreduciblity is a group invariant.

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1 Geometry of free-by-cyclic groups

Given a homeomorphism of a closed surface $f : S \to S$, the mapping torus M_f is defined as $M_f = S \times [0,1]/\sim$ with the equivalence relation: $(x,1) \sim (f(x),0)$ for all $x \in S$. The mapping torus is a 3-manifold whose homeomorphism class is well-defined for the isotopy class $[f]$. Thurston's hyperbolization theorem states that the mapping torus M_f admits an $\mathbb{H}^3$-structure if and only if the monodromy $[f]$ is pseudo-Anosov, i.e. it has no periodic homotopy classes of nontrivial closed curves in S [13, 14]. One consequence of Thurston's hyperbolization theorem is that if the fundamental groups $\pi_1(M_f), \pi_1(M_g)$ are *quasi-isometric*, then the monodromy $[f]$ is pseudo-Anosov if and only if $[g]$ is too; in other words, having pseudo-Anosov monodromy is a quasi-isometry (q.i.) invariant of the mapping torus.

Analogously, given an automorphism of a finite rank free group $\phi : F \to F$, the mapping torus $F \rtimes_\phi \mathbb{Z}$, also called a free-by-cyclic group, is defined by

$$F \rtimes_\phi \mathbb{Z} = \left\langle\, F, t \mid t^{-1}xt = \phi(x), \forall x \in F \,\right\rangle.$$

The isomorphism class of the free-by-cyclic group is well-defined for the outer automorphism $[\phi]$. Brinkmann proved a theorem analogous to Thurston's hyperbolization theorem: the mapping torus $F \rtimes_\phi \mathbb{Z}$ is *word-hyperbolic*, i.e. it satisfies a linear isoperimetric inequality, if and only if the monodromy $[\phi]$ is atoroidal, i.e. it has no periodic conjugacy classes of nontrivial elements in F [4]. As a corollary, having atoroidal monodromy is a q.i. invariant of the free-by-cyclic group. Lately, I have been searching for new q.i. invariants. Specifically, I want to answer the following:

Question. Is having an irreducible monodromy a q.i. invariant of word-hyperbolic free-by-cyclic groups?

An outer automorphism $[\phi]$ is irreducible if it has no invariant free factor system of F. As a first step, I have shown that the property is a *commensurability* invariant:

Theorem 1.1 (cf. [11, Theorem 4.5]). *Suppose $[\phi], [\phi']$ are atoroidal outer automorphisms of free groups and $F \rtimes_\phi \mathbb{Z}, F' \rtimes_{\phi'} \mathbb{Z}$ are commensurable, i.e. they have isomorphic subgroups of finite index. Then $[\phi]$ is irreducible if and only if $[\phi']$ is irreducible.*

In fact, the cited theorem applies to not just automorphisms but injective endomorphisms in general. I only state it for automorphisms here for simplicity.

Whether irreducibility is even a group invariant was an open question asked and partially answered by Dowdall–Kapovich–Leininger (see Question 1.4 and Theorem 1.2 in [5]). The key to my proof was using Feighn–Handel's *preferred presentations* [6] to give an algebraic characterization of when the monodromy is irreducible and atoroidal:

Theorem 1.2 (cf. [11, Theorem 4.3]). *Suppose $\phi : F \to F$ is a free group automorphism. Then $[\phi]$ is irreducible and atoroidal if and only if every finitely generated noncyclic subgroup of $F \rtimes_\phi \mathbb{Z}$ with vanishing Euler characteristic has finite index.*

It seems that a new *geometric structure*, finer than just word-hyperbolicity, is needed to geometrically distinguish irreducible atoroidal monodromies from reducible ones; another approach is to study the *quasi-symmetric* structure on the Gromov boundary $\partial(F \rtimes_\phi \mathbb{Z})$ when the mapping torus is word-hyperbolic since this structure is a complete q.i. invariant. Very little has been done in either approach and the field is wide open.

2 Dynamics of free group automorphisms

With an eye towards geometric structures, I started thinking about algebraic structures. One way to get a canonical algebraic decomposition of $F \rtimes_\phi \mathbb{Z}$ starts by finding a canonical algebraic decomposition of F that is preserved by $[\phi]$. A better understanding of the dynamics of $[\phi]$ is crucial to following this line of thought.

Improved relative train tracks, introduced by Bestvina–Feighn–Handel in [2], have been an invaluable tool for studying outer automorphisms of free groups. Unfortunately, these topological representatives are typically far from unique, which makes it difficult to define canonical invariant decompositions. By leaving the world of graphs (and topological representatives) and working with $\mathbb{R}$-trees instead, I developed canonical representatives for outer automorphisms:

Theorem 2.1 (cf. [12, Main Theorem])**.** *Let $\phi\colon F \to F$ be a free group automorphism. If $[\phi]$ is atoroidal and Λ a $[\phi]$-orbit of its maximal* attracting laminations*, then there is:*

1. *an $\mathbb{R}$-tree (Y_Λ, δ) with a minimal isometric F-action whose arc stabilizers are trivial;*
2. *a unique ϕ-equivariant expanding homothety $h\colon (Y_\Lambda, \delta) \to (Y_\Lambda, \delta)$; and*
3. *an element in F is Y_Λ-loxodromic if and only if its forward ϕ-iterates* limit *to Λ;*

moreover, the tree (Y_Λ, δ) is unique up to rescaling the metric δ.

Again, the cited theorem is more general but this simplified version is enough for what follows. Without defining the terms, I note that atoroidal outer automorphisms have associated finite sets of attracting laminations that are partially ordered by containment.

Corollary 2.2. *Let $\phi\colon F \to F$ be a free group automorphism. If $[\phi]$ is atoroidal and Λ a $[\phi]$-orbit of its attracting laminations, then there is a unique maximal subgroup system $\mathcal{H}_\Lambda$ that is $[\phi]$-invariant and* supports *only attracting laminations contained in Λ.*

As Λ ranges over the $[\phi]$-orbits of attracting laminations, the subgroup systems $\mathcal{H}_\Lambda$ determine a $[\phi]$-invariant canonical algebraic decomposition of F. Their mapping tori, denoted $\mathcal{H}_\Lambda \rtimes_\phi \mathbb{Z}$, determine an algebraic decomposition of the free-by-cyclic group $F \rtimes_\phi \mathbb{Z}$. I am currently working on showing this decomposition is canonical:

Goal 1. *If $[\phi], [\phi']$ are atoroidal outer automorphisms and $F \rtimes_\phi \mathbb{Z} \cong F' \rtimes_{\phi'} \mathbb{Z}$, then $[\phi], [\phi']$ determine the same algebraic decomposition on the free-by-cyclic groups.*

Ultimately, the geometry of the pieces $\mathcal{H}_\Lambda \rtimes_\phi \mathbb{Z}$ should lead to new q.i. invariants of $F \rtimes_\phi \mathbb{Z}$.

3 Beyond free-by-cyclic groups

In my thesis, I extended Brinkmann's theorem to injective endomorphisms $\phi : F \to F$. When the endomorphism is not surjective, the mapping torus is no longer a semi-direct product but the same presentation defines the ascending HNN extension $F*_\phi$:

$$F*_\phi = \left\langle\, F, t \mid t^{-1}xt = \phi(x), \forall x \in F \,\right\rangle .$$

When $F = \mathbb{Z}$ and ϕ is multiplication by $d \neq 0$, then $F*_\phi$ is also denoted by $BS(1,d)$ and called a Baumslag–Solitar group. Like $\mathbb{Z}^2 \cong BS(1,1)$, subgroups isomorphic to $BS(1,d)$ ($d \geq 1$) are obstructions to word-hyperbolicity. It turns out that they are the only obstructions for the mapping torus $F*_\phi$:

Theorem 3.1 (cf. [10, Theorem 5.2.7])**.** *Suppose $\phi : F \to F$ is an injective endomorphism. Then the following are equivalent:*

1. *$F*_\phi$ is word-hyperbolic;*
2. *$F*_\phi$ has no subgroups isomorphic to $BS(1,d)$ for $d \geq 1$; and*
3. *There are no $k, d \geq 1$, $x \in F$, and nontrivial $g \in F$ such that $\phi^k(g) = xg^dx^{-1}$.*

This theorem answers *Gromov's question* in the affirmative for the class of ascending HNN extensions of free groups — the question appears in Mladen Bestvina's GGT problem list:

Question (cf. [1, Question 1.1])**.** Let G be a group of *finite type.* If G has no subgroups isomorphic to $BS(1,d)$ for $d \geq 1$, then must G be word-hyperbolic?

Italiano–Martelli–Migliorini recently constructed a 4-dimensional counterexample to Gromov's question [8]. I suspect hyperbolization theorems are primarily a low-dimensional phenomenon and no lower dimensional counterexample exists. Steve Gersten conjectured an affirmative answer to Gromov's question when restricted to the class of torsion-free one-relator groups — one-relator groups are 2-dimensional. The conjecture has been open for over 20 years and Theorem 3.1 was a step towards its resolution.

I also want to investigate the extent to which the mapping torus $F*_\phi$ can retain *non-positive curvature* properties in general; for instance, I would like to know its optimal isoperimetric inequality. Generally, $F*_\phi$ satisfies an exponential isoperimetric inequality; this inequality is sharp if $F*_\phi$ has a subgroup isomorphic to $BS(1,d)$ for some $d \geq 2$ [7, 9].

Goal 2 (cf. [9, Problem 6.4])**.** *If $F*_\phi$ has no subgroup isomorphic to $BS(1,d)$ for $d \geq 2$, then it satisfies a quadratic isoperimetric inequality.*

Bridson–Groves proved this for free-by-cyclic groups [3].

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