RESEARCH PROSPECTUS

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My area of research is in *Geometric Group Theory* (GGT) — this field is focused on understanding groups as geometric objects and it lies at the intersection of group theory, topology, geometry, dynamics, and more... GGT has proven to be especially useful in the study of low-dimensional topology following William Thurston's seminal work on surfaces and 3-manifolds. In many ways, free groups behave like surface groups and the techniques that were developed to study mapping class groups $\mathrm{MCG}(S)$ have inspired the study of outer automorphism groups of free groups $\mathrm{Out}(F)$. For example, whereas 3-manifolds that fiber over a circle are used to study mapping classes and vice-versa, *free-by-cyclic* groups $F \rtimes_\phi \mathbb{Z}$ are used to understand free group automorphisms $\phi : F \to F$ and vice-versa; Bestvina-Handel defined *irreducible* free group automorphisms as natural analogues for *pseudo-Anosov* mapping classes and used *train tracks* to study their dynamical properties just as Thurston did for pseudo-Anosovs; finally, the Culler-Vogtmann outer space $CV(F)$, the space of marked metric graphs, is the free group analogue of Teichmüller space $\mathcal{T}(S)$, the space of marked hyperbolic surfaces, and these spaces serve as geometric models for $\mathrm{Out}(F)$ and $\mathrm{MCG}(S)$.

However, there is a crucial way this analogy breaks down: surface groups are *coHopfian*, i.e., any injective endomorphism of a surface group is surjective. Free groups on the other hand are far from being coHopfian; fortunately, several theorems for free group automorphisms continue to hold for injective nonsurjective endomorphisms but the lack of an inverse usually complicates the picture. One aspect of my research is understanding dynamical behavior unique to nonsurjective endomorphisms.

There is a second motivation for studying injective free group endomorphisms. Their mapping tori $F*_\phi$, known as ascending HNN extensions, can appear as the base groups in the *Magnus-Moldavansky hierarchy* for torsion-free 1-relator groups, another fundamental class of groups studied in GGT.

In summary, my research focuses on understanding how the dynamics of injective free group endomorphisms relate to the algebraic and geometric properties of their mapping tori. Along these lines, I have proven:

- (Theorem 2.1) If $\phi : F \to F$ is irreducible and nonsurjective, then $F*_\phi$ is word-hyperbolic.
- (Theorem 3.1) The property of ϕ being irreducible and atoroidal is a commensurability invariant of $F*_\phi$.

The second statement answers a question asked by Dowdall-Kapovich-Leininger [7].

1. Dehn functions of $F{*_\phi}$

Given a homeomorphism of a finite type surface $f : S \to S$, the **mapping torus** M_f is defined as $M_f = S \times [0,1]/\sim$ with the equivalence $(x,1) \sim (f(x),0)$ for all $x \in S$. The mapping torus is a 3-manifold that fibers over a circle whose homeomorphism type depends only on the isotopy class of f. Thurston's hyperbolization theorem states that the mapping torus M_f admits an $\mathbb{H}^3$-structure if and only if $\pi_1(M_f)$ has no $\mathbb{Z}^2$ subgroups [15].

On the free group side of the analogy, given an automorphism of a finite rank free group $\phi : F \to F$, the mapping torus $F \rtimes_\phi \mathbb{Z}$, also called a **free-by-cyclic** group, is defined by

$$F \rtimes_\phi \mathbb{Z} = \left\langle\, F, t \mid t^{-1}xt = \phi(x), \forall x \in F \,\right\rangle.$$

The isomorphism type of the free-by-cyclic group depends only on the outer automorphism class of the **monodromy** ϕ. There is a theorem for free group automorphisms due to Brinkmann that is analogous to Thurston's hyperbolization theorem: $F \rtimes_\phi \mathbb{Z}$ is *word-hyperbolic*, i.e., it has a linear *Dehn function* if and only if it has no $\mathbb{Z}^2$ subgroups [6].

I proved a partial generalization of this theorem for injective nonsurjective endomorphisms $\phi : F \to F$. In this case, the mapping torus is no longer a semi-direct product but the same presentation denotes the **ascending HNN extension** $F{*_\phi}$:

$$F{*_\phi} = \left\langle\, F, t \mid t^{-1}xt = \phi(x), \forall x \in F \,\right\rangle.$$

A nonsurjective injective endomorphism $\phi : F \to F$ can map an element to (a conjugate of) a power of itself. When ϕ is induced by a graph map $f : \Gamma \to \Gamma$, then this means that f could (homotopically) map a loop to a power of itself; I shall write f has an **invariant loop** if this happens. With this terminology, I can now state my result:

Theorem 1.1 ([13, Theorem 6.7])**.** *Let $\phi : F \to F$ be an injective endomorphism induced by a graph immersion $f : \Gamma \to \Gamma$. Then the following are equivalent:*

- *$F{*_\phi}$ is word-hyperbolic.*
- *$F{*_\phi}$ has no $BS(1,d)$ subgroups for $d \geq 1$.*
- *f^k has no nontrivial invariant loops for all $k \geq 1$.*

Ilya Kapovich had previously proven a special case of this theorem [10]; he assumed $f : \Gamma \to \Gamma$ was a graph immersion on the rose and his argument was algebraic. By making the argument topological, I was able to generalize it to all graph immersions and also give an effective algorithm that determines whether $F{*_\phi}$ is word-hyperbolic. The three theorems (mine, Kapovich's, and Brinkmann's) all make use of Bestvina-Feighn's combination theorem for word-hyperbolic groups/spaces [3]. Remarkably, my theorem applies to *most* nonsurjective endomorphisms due to Patrick Reynolds' theorem that irreducible nonsurjective endomorphisms can be represented by graph immersions [14].

Corollary 1.2 ([13, Corollary 7.3 & 7.4])**.** *Suppose $\phi : F \to F$ is an injective endomorphism and either ϕ is irreducible or F has rank 2. Then the following are equivalent:*

- *$F{*_\phi}$ is word-hyperbolic.*
- *$F{*_\phi}$ has no $BS(1,d)$ subgroups for $d \geq 1$.*
- *There are no $k, d \geq 1$, $x \in F$, and nontrivial $g \in F$ such that $\phi^k(g) = xg^dx^{-1}$.*

The goal of my thesis is to prove the same equivalence in the corollary without any extra assumptions on the endomorphism or the free group:

Goal 1. *Relax the hypothesis in the Corollary 1.2 to: ϕ is an injective endomorphism.*

Achieving this goal would answer *Gromov's question* in the affirmative for the class of ascending HNN extensions of free groups. This is the first question in Mladen Bestvina's GGT problem list:

Question ([2, Question 1.1])**.** Let G be a group of *finite type*. If G has no $BS(m, d)$ subgroups for $m, d \geq 1$, then must G be word-hyperbolic?

Steve Gersten conjectured that the answer to Gromov's question is "Yes" when restricted to the class of torsion-free 1-relator groups. This conjecture has been open for over 20 years and Goal 1 would be a step towards its resolution.

I also want to understand to what extent the ascending HNN extension $F*_\phi$ is *nonpositively curved* when it fails to be word-hyperbolic; one way to do this is to study the Dehn function. Generally, $F*_\phi$ is *asynchronously automatic* and so has at most exponential Dehn function [1] and if $F*_\phi$ contains a $BS(1, d)$ subgroup for some $d \geq 2$, then it must have an exponential Dehn function [9, 10]. I am working towards showing that the only other possible Dehn functions are linear and quadratic.

Goal 2 ([10, Problem 6.4])**.** *If $F*_\phi$ has no $BS(1, d)$ subgroup for $d \geq 2$, then it has a linear or quadratic Dehn function.*

Bridson-Groves proved this statement in the case when ϕ is an automorphism [5]. To complete Goal 1 and prove the remaining case of Goal 2, I am developing the theory of train tracks for injective endomorphisms.

2. Relative train tracks for nonsurjective $\phi : F \to F$

Improved relative train tracks, introduced by Bestvina-Feighn-Handel [4], have been an invaluable tool for studying individual outer automorphisms as well as subgroups of $\mathrm{Out}(F)$. However, since their existence is only stated for automorphisms, it is unclear if they exist for all injective endomorphisms. One possible approach to determine whether they do exist is to carefully read Bestvina-Feighn-Handel's paper and note where the existence of an inverse is assumed in the construction of improved relative train tracks. Alternatively, one could develop a theory of train tracks for nonsurjective endomorphisms that takes advantage of *nonsurjectiveness* — surprisingly, in a certain sense, nonsurjective endomorphisms tend to have simpler dynamics. In current work-in-progress, I am taking the latter approach with very promising results.

Goal 3. *Construct relative train tracks tailored for nonsurjective endomorphisms.*

The initial steps in this direction were taken by Patrick Reynolds when he proved that irreducible nonsurjective endomorphisms of free groups can be represented by graph immersions [14, Corollary 5.5]. This implies that the dynamics of irreducible nonsurjective endomorphisms are rather simple, which is surprising since, in the surjective case, only finite-order outer automorphisms can be represented by immersions. In my recent paper [12], I give a new proof of Reynolds' theorem that makes use of the semi-action of an endomorphism on outer space $CV(F)$. The main result of the same paper is the following:

Theorem 2.1 ([12, Theorem 6.3])**.** *If $f : \Gamma \to \Gamma$ is an irreducible graph immersion with connected Whitehead graphs that induces $\phi = f_* : F \to F$, then $F{*_\phi}$ is word-hyperbolic.*

Word-hyperbolicity of $F{*_\phi}$ follows from Theorem 1.1 and a characterization of finitely generated subgroups that are invariant under iteration of an irreducible and atoroidal endomorphism:

Proposition 2.2 ([12, Proposition 5.3])**.** *Suppose $\phi : F \to F$ is an irreducible and atoroidal endomorphism and $H \leq F$ is nontrivial and finitely generated. If $i_x\phi^n(H) \leq H$ for some $n \geq 1$ and inner automorphisms i_x, then $[F : (i_x\phi^n)^{-k}(H)] < \infty$ for some $k \geq 0$.*

I am currently working on extending the arguments in [12] to reducible endomorphisms. Roughly speaking, (nonsurjective) injective endomorphisms have canonical fixed (proper) free factor systems relative to which the endomorphisms can be represented by immersions; thus, nonsurjective endomorphisms can be represented by a combination of improved relative train tracks and immersions. With such a structure theorem established, all the goals stated so far readily follow; the next step would be trying to prove Gersten's conjecture. The following covers many base-cases for the Magnus-Moldavansky hierarchies:

Goal 4. *Answer Gromov's question in the affirmative for HNN extensions of finite rank free groups over free factors.*

3. Invariants of $F \rtimes_\phi \mathbb{Z}$

One consequence of Thurston's hyperbolization theorem is that if two surface homeomorphisms f, g have *quasi-isometric* mapping tori M_f, M_g, then one is isotopic to a pseudo-Anosov if and only if the other one is too. This means that the property of the mapping torus M_f having a pseudo-Anosov monodromy is a *geometric invariant*. Similarly, Brinkmann's theorem implies the property of a free-by-cyclic group having an atoroidal monodromy is a geometric invariant and Goal 1 would imply the property of an ascending HNN extension having a monodromy whose iterates have no invariant nontrivial conjugacy classes is a geometric invariant. Lately, I have been interested in finding other geometric invariants. Specifically, I want to answer the following question:

Question. Is having an irreducible and atoroidal monodromy a geometric invariant of ascending HNN extensions of free groups?

I have shown that the property is a *commensurability invariant*:

Theorem 3.1 ([11, Theorem 4.5])**.** *Suppose ϕ, ϕ' are injective endomorphisms whose images are not contained in proper free factors* (minimal) *and $F{*_\phi}, F'{*_{\phi'}}$ are commensurable, i.e., they have isomorphic finite index subgroups. Then ϕ is irreducible and atoroidal if and only if ψ is irreducible and atoroidal.*

That the property is a group invariant was an open question asked and partially answered by Dowdall-Kapovich-Leininger [7, Question 1.4, Theorem 1.2]. The key to my proof was using Feighn-Handel's *preferred presentations* [8] to give an algebraic characterization of $F*_\phi$ that determines exactly when the monodromy is irreducible and atoroidal:

Theorem 3.2 ([11, Theorem 4.3]). *Suppose $\phi : F \to F$ is a minimal injective endomorphism. Then ϕ is irreducible and atoroidal if and only if every finitely generated noncyclic subgroup of $F*_\phi$ with vanishing Euler characteristic has finite index.*

It seems that a new *geometric structure*, possibly stronger than word-hyperbolicity, is needed to geometrically distinguish irreducible endomorphisms from reducible ones; another approach is to understand the *quasi-symmetric* structure on $\partial F*_\phi$ when the mapping torus is word-hyperbolic since this structure is a complete geometric invariant. Very little has been done in either approach and the field is wide open.

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