Relative expansions for free group endomorphisms

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October 9, 2024

Abstract

This survey is based on a minicourse I taught at UC-Riverside in March 2023. The goal is to show the relation between the dynamics of a free group endomorphism and the geometry of its mapping torus. As a new result, we characterize the Dehn functions of ascending HNN extensions of free groups: they are linear, quadratic, or exponential.

The minicourse consisted of four lectures, and this survey's four sections respect that structure (for the most part). The main reference was my thesis [Mut21].

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1 Outline

1.1 Motivation: surface bundles over the circle

The story starts with Bill Thurston's hyperbolization theorem. Suppose Σ is a closed surface with negative Euler characteristic, $f: \Sigma \to \Sigma$ a homeomorphism, and $\varphi = [f]$ the isotopy class of f. The (topological) <u>mapping torus</u> $M_f = \Sigma \times [0,1]/\sim_f$ of f is the quotient space defined using the equivalence relation $(s,0) \sim_f (f(s),1)$ for all $s \in \Sigma$. The 3-manifolds M_f and M_g are homeomorphic if f, g are isotopic; therefore, it makes sense to call them the mapping torus of φ and denote them by M_{φ} . Thurston's remarkable theorem related the dynamics of the mapping class φ to the geometry of its mapping torus M_{φ} :

Theorem 1.1 (cf. [Thu82, Thm. 5.6]). The following are equivalent:

- 1. the mapping torus M_{φ} admits an \mathbb{H}^3 -structure, i.e. there is a homeomorphism from the universal cover of M_{φ} to \mathbb{H}^3 that conjugates the deck transformations to isometries;
- 2. the fundamental group $\pi_1(M_{\varphi})$ has no \mathbb{Z}^2 -subgroup; and
- 3. the mapping class φ is <u>atoroidal</u>: for all $n \ge 1$ and π_1 -injective maps $\sigma \colon \mathbb{S}^1 \to \Sigma$, the maps $f^n \circ \sigma, \sigma$ are not homotopic.

The theorem gives an equivalence of: (1) a geometric property of M_{φ} ; (2) an algebraic property of $\pi_1(M_{\varphi})$; and (3) a dynamical property of φ . The real content (and most difficult part) of this theorem is the implication (3 \Rightarrow 1). It requires a fine understanding of the dynamics of atoroidal mapping classes acting on homotopy classes of closed curves in Σ (and much more). The following theorem is the initial step in this direction and forms part of the Nielsen–Thurston classification [Thu88, Thm. 4] for mapping classes:

Theorem 1.2. The mapping class φ is atoroidal if and only if it contains a pseudo-Anosov homeomorphism of Σ .

We refer the reader to case (ii) of [Thu88, Thm. 4] for the definition of pseudo-Anosov homeomorphisms. Some authors will call a mapping class pseudo-Anosov if it contains a pseudo-Anosov homeomorphism. For our purposes, we mention that the mapping class containing a pseudo-Anosov homeomorphism allows Thurston to apply the double limit theorem, a separate deep result [Thu82, Thm. 5.4], to conclude Theorem 1.1.

1.2 Motivation: free-by-cyclic groups

Bringing the story closer to our goal, Peter Brinkmann proved a (coarse) hyperbolization theorem analogous to Theorem 1.1. Let \mathbb{F} be a finitely generated free group, $\Phi \colon \mathbb{F} \to \mathbb{F}$ an automorphism, and $\phi = [\Phi]$ the outer automorphism for Φ . The (algebraic) <u>mapping torus</u> $\mathbb{F} \rtimes_{\Phi} \mathbb{Z}$ of Φ is the free-by-cyclic group given by the relative presentation:

$$\mathbb{F} \rtimes_{\Phi} \mathbb{Z} = \langle \mathbb{F}, t \mid txt^{-1} = \Phi(x), \ \forall x \in \mathbb{F} \rangle;$$

as before, the free-by-cyclic groups $\mathbb{F} \rtimes_{\Phi} \mathbb{Z}$ and $\mathbb{F} \rtimes_{\Psi} \mathbb{Z}$ are isomorphic for any $\Psi \in \phi$, and we denote them by $\mathbb{F} \rtimes_{\phi} \mathbb{Z}$. Brinkmann's theorem related the dynamics of an outer automorphism to the geometry of its mapping torus:

Theorem 1.3 (cf. [Bri00, Thm. 1.2]). The following are equivalent:

- 1. the mapping torus $\mathbb{F} \rtimes_{\phi} \mathbb{Z}$ is hyperbolic;
- 2. $\mathbb{F} \rtimes_{\phi} \mathbb{Z}$ has no \mathbb{Z}^2 -subgroup; and
- 3. the outer automorphism ϕ is <u>atoroidal</u>: for all $n \ge 1$ and nontrivial $x \in \mathbb{F}$, the elements $\Phi^n(x), x$ are not conjugate in \mathbb{F} .

See Section 4 for a definition of hyperbolic groups. As with Thurston's theorem, Brinkmann's theorem gives an equivalence of geometric, algebraic, and dynamical properties, and the hard part is the implication $(3\Rightarrow1)$. The outer automorphism ϕ is <u>hyperbolic</u> if there is a constant $L \ge 1$ such that $2 \cdot ||\Phi^L(x)|| \le \max(||x||, ||\Phi^{2L}(x)||)$ for all $x \in \mathbb{F}$; here, ||x|| is the word-length of the shortest representative of the conjugacy class of x in \mathbb{F} with respect to some arbitrarily chosen basis of \mathbb{F} . Brinkmann's contribution was the following intermediate step:

Theorem 1.4 (cf. [Bri00, Thm. 1.1]). If the outer automorphism ϕ is atoroidal, then it is hyperbolic.

Theorem 1.3 then follows from a special case of Bestvina–Feighn's combination theorem [BF92]: hyperbolicity of an outer automorphism implies hyperbolicity of its mapping torus.

1.3 Objective: ascending HNN extensions of free groups

To conclude our story, we extended Brinkmann's theorem to the case when $\Phi \colon \mathbb{F} \to \mathbb{F}$ is an injective but not necessarily surjective endomorphism. The <u>outer class</u> $\phi = [\Phi]$ is the set of endomorphisms of \mathbb{F} obtained by post-composing Φ with the inner automorphisms of \mathbb{F} . The <u>mapping torus</u> $\mathbb{F}*_{\phi}$ of ϕ is the ascending HNN extension of \mathbb{F} given by the same relative presentation as before:

$$\mathbb{F}*_{\phi} = \langle \mathbb{F}, t \mid txt^{-1} = \Phi(x), \ \forall x \in \mathbb{F} \rangle;$$

the only difference is that \mathbb{F} is not a normal subgroup of the mapping torus if Φ is not surjective. See [Ser77, §I.1.4] for the general definition of HNN extensions. If $\mathbb{F} = \mathbb{Z}$, then Φ is multiplication by a nonzero integer d and its mapping torus, denoted BS(1, d), is an example of a Baumslag–Solitar group; note that $BS(1, 1) = \mathbb{Z}^2$ and BS(1, -1) are the fundamental groups of the torus and Klein bottle respectively. In this survey, we sketch an equivalence between the dynamics of ϕ and the geometry of $\mathbb{F}*_{\phi}$.

Theorem 4.3 (cf. [Mut21, Thm. 5.2.7]). The following are equivalent:

- 1. The mapping torus \mathbb{F}_{ϕ} is hyperbolic;
- 2. \mathbb{F}_{ϕ} has no BS(1, d)-subgroup for all $d \geq 1$; and
- 3. for all $n, d \ge 1$ and nontrivial $x \in \mathbb{F}$, the elements $\Phi^n(x), x^d$ are not conjugate in \mathbb{F} .

Condition (3) is stronger than the previous atoroidal property. To prove the difficult implication $(3\Rightarrow1)$, we will establish some quantitative properties — hyperbolicity of the outer endomorphism ϕ is one of them but is not sufficient when the endomorphism Φ is not surjective. The implication will then follow from Bestvina–Feighn's combination theorem.

Misha Gromov proved the equivalence between hyperbolicity of a finitely presented group and linearity of its Dehn function (see Section 4 for definitions). Bridson–Groves show that $\mathbb{F} \rtimes_{\phi} \mathbb{Z}$ has a linear or quadratic Dehn function [BG10]. Ilya Kapovich observed that the Dehn function of \mathbb{F}_{ϕ} is exponential if it contains a BS(1, d)-subgroup for some $d \geq 2$ [Kap00, Cor. 5.7]. As a new result, we use relative hyperbolicity to answer Kapovich's question asking for all the possible Dehn functions [Kap00, Prob. 6.5]:

Corollary 4.8. Ascending HNN extensions of finitely generated free groups have linear, quadratic, or exponential Dehn functions.

1.4 Prerequisites

Familiarity with basic algebraic topology will be assumed; for instance, we consider CWcomplexes, fundamental groups, and covering spaces. We also assume familiarity with John Stallings' paper "Topology of finite graphs" [Sta83]. While we define the graph-centric terms (e.g. folds, pullbacks,...) in our notes, we do not provide illustrations and examples; Stallings' paper is a short and perfect reference for this. We use the language of Bass–Serre theory in Section 4; the first half of Jean-Pierre Serre's book *Trees* will be our reference [Ser77, Chapter I. Trees and Amalgams]. Finally, the reader may use the notes [ABC⁺91] as a standard reference for hyperbolic groups, and the papers [Far98, Bow12] for relatively hyperbolic groups.

Acknowledgments. I thank: Matthew Durham and Thomas Koberda for organizing the workshop and inviting me to give a minicourse; Kailey Perry for giving the introductory talk for the minicourse; and Jagerynn Verano (and anonymous referees) for helpful comments on a preliminary version.

2 Expansions of free splittings

Throughout the survey, \mathbb{F} will denote a finitely generated nontrivial free group, $\Phi \colon \mathbb{F} \to \mathbb{F}$ an injective endomorphism, and $\phi = [\Phi]$ its outer class.

The hyperbolization theorems (Theorems 1.3 and 4.3) need a fine understanding of the dynamics of ϕ acting on conjugacy classes of elements in \mathbb{F} . When Φ is surjective, the theory of train tracks is a robust framework for studying these dynamics. In particular, Bestvina–Feighn–Handel introduced improved relative train tracks in [BFH00], and Brinkmann used them to prove Theorem 1.4.

We mostly care about the case when Φ is not surjective. Our current goal is to find optimal representatives that exhibit the needed dynamics of ϕ . Rather than try to directly extend the theory of improved relative train tracks to the nonsurjective case, we will develop our optimal representatives "from scratch" using nonsurjectivity to our advantage. No knowledge of train tracks is necessary or assumed. We will use Stallings maps, bounded cancellation, and free splittings.

2.1 Stallings maps

A graph is a 1-dimensional CW-complex, and the volume $\operatorname{vol}(\Gamma)$ of a finite graph Γ is the number of edges in Γ . A core graph is a graph whose components are not contractible and have no proper deformation retracts. The core of a graph is the smallest deformation retract of the union of the graphs noncontractible components. A cellular map is a continuous function between graphs that maps vertices to vertices and is locally injective or constant on edges. An immersion is a cellular map that is locally injective everywhere. An expansion is an immersion $g: \Gamma \to \Gamma$ such that the combinatorial length (i.e. number of edges) of the edge-paths $g^n(e)$ is unbounded as $n \to \infty$ for every edge e in Γ .

Example 2.1. Let $\mathbb{S}^1 = \{z \in \mathbb{C} : ||z|| = 1\}$; define a CW-structure by declaring $1 \in \mathbb{S}^1$ to be a vertex and $\mathbb{S}^1 \setminus \{1\}$ an edge. This core graph is a rose (i.e. a graph with one vertex) with one petal (i.e. edge). For an integer $d \neq 0$, the cellular map $\mathbb{S}^1 \to \mathbb{S}^1$ given by $z \mapsto z^d$ is an immersion that is an expansion if and only if $|d| \geq 2$.

There are three basic cellular maps that will be instrumental to our constructions of immersions and expansions. Fix a graph Γ , edges e_1, e_2 in Γ , and a point $p \in e_1$. We define a new CW-structure Γ' on the underlying space of Γ : declare p to be a vertex (in addition to the vertices of Γ) and replace e_1 with the two components of $e_1 \setminus \{p\}$. The identity map on the underlying space is a cellular map $\Gamma \to \Gamma'$ known as a <u>subdivision</u>. Define an equivalence relation \sim_2 on Γ by setting $x \sim_2 y$ if x = y or x, y are in the closure of e_2 . The quotient $\Gamma'' = \Gamma/\sim_2$ inherits a natural CW-structure such that the quotient map $\Gamma \to \Gamma''$ is a cellular map known as an <u>edge collapse</u>. Finally, assume e_1, e_2 are distinct. We represent orientations of the edges as topological embeddings $\epsilon_1, \epsilon_2: (0, 1) \to \Gamma$ of the open unit interval into the underlying space of Γ . If the continuous extensions $\bar{\epsilon}_1, \bar{\epsilon}_2: [0, 1] \to \Gamma$ satisfy $\overline{\epsilon}_1(0) = \overline{\epsilon}_2(0)$, then we can define another equivalence relation \sim_f on Γ by setting $x \sim_f y$ if x = y or there is some $t \in [0, 1]$ such that $x = \overline{\epsilon}_1(t)$ and $y = \overline{\epsilon}_2(t)$. The quotient $\Gamma''' = \Gamma/\sim_f$ inherits a natural CW-structure such that the quotient map $\Gamma \to \Gamma'''$ is a cellular map known as a fold (of oriented edges). The following theorem is due to Stallings.

Theorem 2.1. Any cellular map of finite graphs is a composition of subdivisions, edge collapses, folds, and an immersion that maps edges to edges.

We refer to this as the Stallings decomposition of the cellular map.

Proof. It is immediate from the definitions that any cellular map of finite graphs is a composition of subdivisions, edge collapses, and a cellular map that maps edges to edges. Stallings proved that any cellular map of finite graphs that maps edges to edges is a composition of folds and an immersion [Sta83, §3.3]. \Box

Let Γ be a connected finite core graph and $\star \in \Gamma$ a chosen vertex. Assume Γ is a <u>marked graph</u>, i.e. we have fixed an identification/marking $\pi_1(\Gamma, \star) \cong \mathbb{F}$. Pick a nontrivial subgroup $A \leq \mathbb{F}$, and let $c: (\widehat{\Gamma}, \widehat{\star}) \to (\Gamma, \star)$ be the based cover corresponding to A. The <u>Stallings based map</u> $\widehat{s}: (\widehat{S}, \widehat{\star}) \to (\Gamma, \star)$ for A over (Γ, \star) is the restriction of c to the smallest deformation retract of $(\widehat{\Gamma}, \widehat{\star})$ containing $\widehat{\star}$; there is an algorithm for constructing \widehat{s} when A is finitely generated [Sta83, §5.4]. The <u>Stallings map</u> $s: S \to \Gamma$ for the conjugacy class [A] (over Γ) is the restriction of \widehat{s} to the core of $\widehat{\Gamma}$. For $n \geq 1$, let $s_n: S(\phi^n) \to \Gamma$ denote the Stallings map for $[\Phi^n(\mathbb{F})]$.

Example 2.2. Suppose $\mathbb{F} = \mathbb{Z}$. Then Φ is multiplication by $d \neq 0$. We identify \mathbb{Z} with the fundamental group $\pi(\mathbb{S}^1, \star)$ of the rose with one petal. For $n \geq 1$, the Stallings map $s_n \colon S(\phi^n) \to \mathbb{S}^1$ is a cover with $|d|^n$ sheets; as a CW-complex, $S(\phi^n)$ has $|d|^n$ vertices and edges. Note that $\operatorname{vol}(S(\phi^n)) \to \infty$ as $n \to \infty$ if and only if Φ is not surjective (i.e. $|d| \geq 2$).

When Φ is surjective, the Stallings maps s_n are the identity map and not really useful for anything. On the other hand, the previous example illustrates that they get arbitrarily complicated (as measured with volume) when \mathbb{F} is cyclic and Φ is not surjective. The following lemma shows that the dichotomy holds even when \mathbb{F} is not cyclic:

Lemma 2.2. Let Γ be a marked graph and $S(\phi^n) \to \Gamma$ the Stallings map for $[\Phi^n(\mathbb{F})]$. The endomorphism Φ is not surjective if and only if $\operatorname{vol}(S(\phi^n)) \to \infty$ as $n \to \infty$.

Proof. If Φ is surjective, then s_n is the identity id: $\Gamma \to \Gamma$, and $\operatorname{vol}(S(\phi^n)) = \operatorname{vol}(\Gamma)$ is constant in $n \geq 1$. Conversely, suppose there exists a constant $V \geq 1$ such that $\operatorname{vol}(S(\phi^n)) \leq V$ for infinitely many $n \geq 1$. There are only finitely many combinatorially distinct immersions $\Gamma' \to \Gamma$ (mapping edges to edges) with $\operatorname{vol}(\Gamma') \leq V$. So, for some $n' > n \geq 1$, there is a cellular isomorphism $\iota: S(\phi^{n'}) \to S(\phi^n)$ such that $s_{n'} = s_n \circ \iota$. This can be extended to an isomorphism $\hat{\iota}: \widehat{\Gamma}_{n'} \to \widehat{\Gamma}_n$ of covers of Γ . Correspondence between covers and conjugacy classes of subgroups implies $[\Phi^{n'}(\mathbb{F})] = [\Phi^n(\mathbb{F})]$. In other words, $x\Phi^n(\mathbb{F})x^{-1} = \Phi^{n'}(\mathbb{F}) \leq \Phi^n(\mathbb{F})$ for some $x \in \mathbb{F}$. It is as a neat exercise that $x\Phi^n(\mathbb{F})x^{-1} = \Phi^n(\mathbb{F})$ and hence Φ is surjective. \Box

The exercise at the end of the proof is a special case of the following:

Exercise 2.3. Let $\Psi : \mathbb{F} \to \mathbb{F}$ be an automorphism and $A \leq \mathbb{F}$ a finitely generated subgroup. If $\Psi(A) \leq A$, then $\Psi(A) = A$.

Hint. Use subgroup separability (also known as the LERF-property or Hall's theorem): for any finitely generated subgroup $A \leq \mathbb{F}$ and element $b \in \mathbb{F} \setminus A$, there is a homomorphism $\pi \colon \mathbb{F} \to F$ to a finite group F such that $\pi(b) \notin \pi(A)$; moreover, subgroup separability can be proven using the Stallings based map for A over a rose [Sta83, §6].

2.2 Bounded cancellation

Let Γ be a marked graph. A cellular map $g \colon \Gamma \to \Gamma$ represents ϕ if g induces ϕ via the chosen marking $\pi_1(\Gamma) \cong \mathbb{F}$. The outer class ϕ is reducible if it can be represented by a cellular map with an invariant nonempty proper core subgraph; otherwise, it is <u>irreducible</u>.

Example 2.4. Suppose $\mathbb{F} = \langle a, b \rangle$ and $\Phi \colon \mathbb{F} \to \mathbb{F}$ maps $a \mapsto a$ and $b \mapsto ba^{-1}b^{-1}$. Let R be the oriented rose with two petals, and pick the marking $\pi_1(R) \cong \mathbb{F}$ that identifies the basis $\{a, b\} \subset \mathbb{F}$ with the petals of R. We abuse notation and refer to the petals by a, b accordingly (see Fig. 1); the petals with opposite orientations are \bar{a}, \bar{b} . The cellular map $f \colon R \to R$ that maps a to a and b to the edge-path $b\bar{a}\bar{b}$ represents the outer class $\phi = [\Phi]$. The petal a determines an f-invariant proper core subgroup of R, and so ϕ is reducible.

Now suppose $\Psi \colon \mathbb{F} \to \mathbb{F}$ maps $a \mapsto ab$ and $b \mapsto ba$, and let $\psi = [\Psi]$ be the outer class. It takes more work to show that an outer endomorphism is irreducible. As \mathbb{F} has rank 2, any nonempty proper core subgraph of an \mathbb{F} -marked graph has circle components; therefore, to prove that ψ is irreducible, it suffices to show that $\Psi^n(x)$ and x^d are not conjugate in \mathbb{F} for all $n, d \geq 1$ and nontrivial $x \in \mathbb{F}$. For the case d = 1, note that the cellular map $g \colon R \to R$ that maps a to ab and b to ba is an expansion representing ψ (see Fig. 1). The case $d \geq 2$ will follow from Exercises 3.3 and 3.4. We return to these examples throughout the survey.



Figure 1: A marked rose and two cellular maps. We illustrate a cellular map $g: \Gamma \to \Gamma'$ between oriented graphs that is locally injective on edges by overlaying each edge e in Γ with the labels of the immersed edge-path g(e) in Γ' .

Part of Patrick Reynolds' thesis [Rey11] shows that ϕ is (uniquely) represented by an expansion if it is irreducible and Φ is not surjective. We give a proof that is comparatively elementary as it only uses the classical bounded cancellation lemma and Stallings maps.

Lemma 2.3 (bounded cancellation). If $f: \Gamma' \to \Gamma$ is a cellular map of finite graphs and $\tilde{f}: \tilde{\Gamma}' \to \tilde{\Gamma}$ a lift to the universal covers, then there is a constant $C \ge 0$ such that the \tilde{f} -image of the geodesic [p,q] is in the C-neighborhood of $[\tilde{f}(p), \tilde{f}(q)]$ for all vertices $p, q \in \tilde{\Gamma}'$.

The minimal such constant C is the <u>cancellation constant</u> for the cellular map f and denoted C(f). In a simplicial tree (i.e. simply connected graph), the geodesic [u, v] between two vertices u, v is the unique immersed edge-path that has the vertices as it endpoints. (Combinatorial) Neighborhoods of edge-paths are considered with respect to the combinatorial length. The following proof is adapted from Bestvina–Feighn–Handel [BFH97, Lem. 3.1].

Proof. By Theorem 2.1, the cellular map f is a composition of subdivisions, edge collapses, n folds for some $n \ge 0$, and an immersion that maps edges to edges. The lemma holds for: subdivisions and edge collapses with C = 0; folds with C = 1; and immersions with C = 0. By the next exercise, the lemma holds for f with C = n.

Exercise 2.5. Suppose Lemma 2.3 holds for cellular maps $g: \Gamma' \to \Gamma$ and $g': \Gamma'' \to \Gamma'$ with constants C and C' respectively. If g' is a subdivision or g maps edges to edges or vertices, then the lemma holds for $g \circ g': \Gamma'' \to \Gamma$ with the constant C + C'.

A point in a graph is <u>bivalent</u> if it has a neighborhood homeomorphic to \mathbb{R} . A point in a core graph is a <u>branch point</u> if it is not bivalent. Let Γ be a core graph with no component homeomorphic to \mathbb{S}^1 . A <u>natural arc</u> of Γ is a maximal connected subset that contains no branch point. The <u>natural structure</u> on Γ is the CW-complex whose vertices are the branch points and edges are the natural arcs (i.e. forget the bivalent vertices). A cellular map $g: \Gamma \to \Gamma$ is <u>natural</u> if it is a cellular map with respect to the natural structure, i.e. it maps branch points to branch points and is injective or constant near bivalent points.

Theorem 2.4 (cf. [Rey11, Cor. 3.23]). If the endomorphism Φ is not surjective and its outer class ϕ is irreducible, then some expansion $g: \Gamma \to \Gamma$ represents ϕ .

Proof (cf. [Mut21, Prop. 3.4.1]). Let rank(\mathbb{F}) ≥ 2 and $g_0: \Gamma_0 \to \Gamma_0$ be a cellular map representing ϕ via a marking $\pi_1(\Gamma_0) \cong \mathbb{F}$. The map g_0 is K-Lipschitz (for some $K \geq 1$) with cancellation constant $C = C(g_0)$. Let $\widehat{\Gamma}_n \to \Gamma_0$ be the cover corresponding to $[\Phi^n(\mathbb{F})]$ and $S(\phi^n) \to \Gamma_0$ the Stallings map for $[\Phi^n(\mathbb{F})]$. The lift of g_0 to $\widehat{\Gamma}_n$ and the deformation retraction $\widehat{\Gamma}_n \to S(\phi^n)$ induce a K-Lipschitz map $g_n: S(\phi^n) \to S(\phi^n)$ with $C(g_n) \leq C$. By bounded cancellation, the map g_n maps branch points to the C-neighborhood of branch points. After replacing g_n with a homotopic map, we may assume it is a natural map with Lipschitz constant K + C and cancellation constant $C(g_n) \leq 2C$.

By nonsurjectivity of Φ and Lemma 2.2, we can pick $N \gg 1$ such that $S(\phi^N)$ has a natural arc with more than $2C(K+C)^M$ edges, where $M = 3 \operatorname{rank}(\mathbb{F}) - 4$. A natural arc α of $S(\phi^N)$ is long if $g_N^m(\alpha)$ covers a natural arc with more than $2C(K+C)^M$ edges for some $m \ge 0$; otherwise, it is *short*. Since $S(\phi^N)$ has at most M+1 natural arcs and g_N

is (K + C)-Lipschitz, long natural arcs have more than 2C edges. The g_N -invariant short subgraph (i.e. closure of union of short natural arcs) is a proper subgraph. By irreducibility of ϕ , the components of the short subgraph are contractible — note that g_N represents ϕ via the isomorphism $\Phi^N \colon \mathbb{F} \to \Phi^N(\mathbb{F})$.

Let Γ be the graph obtained by collapsing in $S(\phi^N)$ each component of the short subgraph. Then the natural map $g: \Gamma \to \Gamma$ induced by g_N represents ϕ . Since the natural arcs of Γ have more than $2C \ge C(g_N) \ge C(g)$ edges, there are no folds in the Stallings decomposition of the natural map g and C(g) = 0. By injectivity of Φ , the subgraphs on which g is constant are contractible; after repeatedly collapsing these subgraphs, we may assume g is an immersion. An edge e of Γ is *non-expanding* if its iterates $g^m(e)$ are edges for all $m \ge 1$. The g-invariant non-expanding subgraph (i.e. closure of the union of non-expanding edges) is proper as Φ is not surjective. By irreducibility of ϕ again, the components of the non-expanding subgraph are contractible. Collapse the non-expanding subgraph to ensure g is an expansion.

By Lemma 2.2, expansions cannot represent outer automorphisms.

Exercise 2.6. If an expansion represents the outer endomorphism ϕ , then the endomorphism Φ is not surjective.

More generally, an expansion represents ϕ if and only if there are no ϕ -periodic (conjugacy classes of) nontrivial free factors of \mathbb{F} — a precise statement is given in the next subsection. By bounded cancellation, an expansion representing ϕ is unique if it exists.

Exercise 2.7. If $g: \Gamma \to \Gamma$ and $g': \Gamma' \to \Gamma'$ are expansions representing the outer endomorphism ϕ , then there is a homeomorphism $h: \Gamma \to \Gamma'$ such that $g' \circ h = h \circ g$.

Hint. This exercise is a good test for the reader's understanding of our proof of Theorem 2.4. Let a cellular map $h_0: \Gamma \to \Gamma'$ be a homotopy equivalence between the marked graphs that induces the trivial outer automorphism of \mathbb{F} . So $g' \circ h_0 \simeq h_0 \circ g$. Denote the Stallings maps for $[\Phi^n(\mathbb{F})]$ over Γ and Γ' by $s_n: S(\phi^n) \to \Gamma$ and $s'_n: S(\phi^n)' \to \Gamma'$ respectively. Since g is an expansion, $S(\phi^n)$ is a subdivision of Γ ; the same holds for $S(\phi^n)'$ and Γ' . By considering natural maps $h_n: S(\phi^n) \to S(\phi^n)'$ that are homotopic to lifts of h_0 for $n \gg 1$, bounded cancellation can be used to deduce that h_0 is homotopic to a homeomorphism.

2.3 Free splittings

We introduce our final tool: free splittings! Let Γ be a finite core graph with no bivalent vertices. A graph pair (Γ, G) is a choice of a proper core subgraph $G \subset \Gamma$, which we call pair's <u>peripheral subgraph</u>. For a graph pair (Γ, G) , edges of G are <u>peripheral</u> and the remaining edges of Γ are <u>nonperipheral</u>. The <u>relative length</u> of an edge-path in Γ is the number of nonperipheral edges in the path. A relative cellular map $(\Gamma, G) \to (\Gamma', G')$ is a cellular map $\Gamma \to \Gamma'$ that maps G to G'. The relative universal cover of (Γ, G) is the result of collapsing in $\widetilde{\Gamma}$ each component of \widetilde{G} , where $\widetilde{\Gamma}$ is the universal cover of Γ and $\widetilde{G} \subset \widetilde{\Gamma}$ the lift of G. A relative immersion is a relative cellular map $(\Gamma, G) \to (\Gamma, G)$ whose lifts to the universal covers induce immersions on the relative universal covers. Finally, a relative expansion is a relative immersion $g: (\Gamma, G) \to$ (Γ, G) such that the relative length of $\overline{g^n(e)}$ is unbounded as $n \to \infty$ for every nonperipheral edge e in Γ . When G is empty, these relative notions are exactly the usual ones.

Example 2.8. Suppose $\mathbb{F} = \langle a, b \rangle$ and $\Phi \colon \mathbb{F} \to \mathbb{F}$ maps $a \mapsto a^{-1}$ and $b \mapsto bab^{-1}$. Let B be the (oriented) barbell graph (see Fig. 2): it consists of two (left/right) loops ℓ, r and a separating (middle) edge m. Pick the marking that identifies a, b with $\ell, mr\bar{m}$ respectively. The cellular map $h \colon B \to B$ that maps $\ell \mapsto \bar{\ell}, m \mapsto mr\bar{m}$, and $r \mapsto \ell$ is an immersion but not an expansion. Now let $G \subset B$ be the proper core subgraph that is the closure of $\ell \cup r$. The relative cellular map $h \colon (B, G) \to (B, G)$ is a relative expansion representing ϕ . The map h preserves the component of G corresponding to ℓ , and the restriction of h to this component is a homotopy equivalence — this motivates the next definitions.



Figure 2: A marked barbell and a relative expansion.

The <u>peripheral restriction</u> of a relative cellular map $g: (\Gamma, G) \to (\Gamma, G)$ is the restriction of \overline{g} to \overline{G} . The <u>g-stable part</u> G' of G is the union of components of G that intersect $g^N(G)$, where N is the number of components in G; the peripheral restriction is an almost homotopy equivalence if its restriction to G' is a homotopy equivalence.

Now assume Γ is also connected. A free splitting (of \mathbb{F}) is a graph pair (Γ , G) with a marking $\pi_1(\Gamma) \cong \mathbb{F}$. A nontrivial conjugacy class [x] is peripheral in a free splitting (Γ , G) if the immersed loop in Γ representing [x] is in G; otherwise, it is nonperipheral.

Remark. In [Mut21], a free splitting is defined as the relative universal cover of a marked graph pair. Relative cellular maps are then the "lifts" of cellular maps of pairs to the relative universal covers; relative immersions/expansions are defined on the relative covers as well. The advantage of this point of view is that relative immersions/expansions are honest immersions/expansions, which makes proofs clearer. The downside is that relative universal covers are infinite objects and hard to illustrate. Generally, a free splitting of a group \mathbb{G} is a simplicial action of the group on a simplicial tree with trivial stabilizers for points in edges; equivalently, a free splitting of \mathbb{G} is a graph of groups decomposition of \mathbb{G} with trivial edge groups [Ser77, §I.5 Thm. 13].

By the same argument used for Exercise 2.6, a relative expansion cannot represent an

outer automorphism. The converse, a generalization of Reynold's Theorem 2.4, is the heart of [Mut21] and this survey.

Theorem 2.5 (cf. [Mut21, Thm. 3.4.5]). If the endomorphism Φ is not surjective, then its outer class ϕ is represented by a relative expansion $g: (\Gamma, G) \to (\Gamma, G)$ whose peripheral restriction is an almost homotopy equivalence.

Sketch of proof. Construct a (K + C)-Lipschitz natural map $g_N \colon S(\phi^N) \to S(\phi^N)$ with cancellation constant $C(g_N) \leq 2C$ and a g_N -invariant short proper subgraph $G_N \subset S(\phi^N)$ as in our proof of Theorem 2.4. This time, Γ is obtained by: collapsing in $S(\phi^N)$ each contractible component of G_N ; deformation retracting the noncontractible components to their cores; and forgetting bivalent vertices. Let $G \subset \Gamma$ be the image of $\operatorname{core}(G_N)$, and call this process "relative collapsing G_N " in $S(\phi^N)$. Then (Γ, G) is a free splitting, and the induced relative cellular map $g \colon (\Gamma, G) \to (\Gamma, G)$ represents ϕ with relative cancellation constant $C_G(g) = 0$, i.e. the induced maps on the relative universal cover has cancellation constant C = 0.

In the base case, the peripheral restriction of g is an almost homotopy equivalence. By injectivity of Φ , core(G') = core(G) for subgraphs $G' \supset G$ on which g is relatively constant. After repeatedly relatively collapsing these subgraphs, we may assume g is a relative immersion. As Φ is not surjective, some edge must be relatively expanding. Relatively collapse the non-relatively-expanding edges to ensure g is a relative expansion (whose peripheral restriction is an almost homotopy equivalence). This concludes the base case.

By induction, we may suppose the peripheral restriction g_G of g is a relative expansion on a graph pair (G, H) whose peripheral restriction is an almost homotopy equivalence. For a simplification of the remaining steps, you may pretend H is empty — the steps in the general relative setting are nearly identical. There are two cases to consider: either some $g^m(\Gamma) \subset G$ for some $m \geq 1$; or g is a relative immersion (after repeatedly relatively collapse subgraphs on which g is relatively constant if necessary).

Case 1: the Stallings map for $[\Phi^m(\mathbb{F})]$ over (G, H) determines a free splitting with a relative expansion representing ϕ whose peripheral restriction is an almost homotopy equivalence. Case 2: use the relative expansion g_G and bounded cancellation to promote g to a relative expansion whose peripheral restriction is an almost homotopy equivalence.

If a free splitting (Γ, G) admits a relative expansion g whose peripheral restriction is an almost homotopy equivalence, then the g-stable part of G determines the unique (up to conjugacy) free factor system consisting of maximal ϕ -periodic nontrivial free factors of \mathbb{F} ; thus any two such free splittings determine the same peripheral conjugacy classes in \mathbb{F} . By bounded cancellation, there is an essentially unique relative expansion representing ϕ whose peripheral restriction is an almost homotopy equivalence [Mut21, Cor. 3.4.7] (when Φ is not surjective), and we call it the <u>canonical relative expansion</u> for ϕ . Note that an expansion represents ϕ if and only if Φ is not surjective and the canonical relative expansion for ϕ is an expansion (see also [Mut21, Cor. 3.4.8]). Our first main application of canonical relative expansions will be the extension of Brinkmann's Theorem 1.4. Pick an arbitrary marked graph Γ . For $x \in \mathbb{F}$, let ||x|| be the combinatorial length of the immersed (or constant) loop in Γ representing the conjugacy class [x] in \mathbb{F} . The outer endomorphism ϕ is <u>hyperbolic</u> if there is a constant $L \geq 1$ such that $2 \cdot ||\Phi^L(x)|| \leq \max(||x||, ||\Phi^{2L}(x)||)$ for all $x \in \mathbb{F}$; this property is independent of the chosen marked graph Γ . The following is a nice exercise for working with the definition of hyperbolicity.

Exercise 2.9. If the outer endomorphism ϕ is represented by an expansion, then it is hyperbolic.

The outer class ϕ is <u>atoroidal</u> if $\Phi^n(x)$ is not conjugate to x in \mathbb{F} for all $n \ge 1$ and nontrivial $x \in \mathbb{F}$. Here is another exercise for working with hyperbolicity.

Exercise 2.10. If the outer endomorphism ϕ is hyperbolic, then it is atoroidal.

The converse for this exercise follows from Theorem 1.4 and the existence of canonical relative expansions.

Proposition 2.6 (cf. [Mut21, Prop. 5.2.6]). If the outer endomorphism ϕ is atoroidal, then it is hyperbolic.

Sketch of proof. Theorem 1.4 covers the case when Φ is surjective. So we may assume Φ is not surjective. Let $g: (\Gamma, G) \to (\Gamma, G)$ be the canonical relative expansion for ϕ . There is a constant $L_0 \geq 1$ such that $2 \cdot ||x||_G \leq ||\Phi^{L_0}(x)||_G$ for all $x \in \mathbb{F}$, where $||x||_G$ is the relative length of the immersed loop in Γ representing [x]. The peripheral restriction is an almost homotopy equivalence and, by Theorem 1.4 again, there is a constant $L_1 \geq 1$ such that $2 \cdot ||\Phi^{L_1}(x)|| \leq \max(||x||, ||\Phi^{2L_1}(x)||)$ for all peripheral $x \in \mathbb{F}$ since ϕ is atoroidal. With care, we can deduce that ϕ is hyperbolic.

2.4 Side quest: stable images

Most of this survey focuses on the outer class ϕ of the injective endomorphism $\Phi \colon \mathbb{F} \to \mathbb{F}$. Let us take a brief excursion into studying Φ .

The stable image of Φ is the intersection $\Phi^{\infty}(\mathbb{F}) = \bigcap_{n\geq 1} \Phi^n(\mathbb{F})$. The stable image is Φ -fixed (setwise), i.e. $\Phi(\Phi^{\infty}(\mathbb{F})) = \Phi^{\infty}(\mathbb{F})$; in particular, the restriction of Φ to the stable image is an automorphism, and Edward Turner proved the stable image is a free factor of \mathbb{F} [Tur96, Thm. 1] (see also the algorithm of Corollary 2.9). It can be used to reduce questions about injective endomorphisms to the corresponding questions about automorphisms. For algorithmic reductions, we would need to compute the stable image.

Let (Γ, \star) be a connected finite graph Γ with a chosen vertex/basepoint $\star \in \Gamma$; it is a pointed core graph if there is no proper deformation retract containing the basepoint. A <u>based cellular map</u> $g: (\Gamma, \star) \to (\Gamma, \star)$ is a cellular map that fixes the basepoint \star ; it represents Φ if there is a marking $\pi_1(\Gamma, \star) \cong \mathbb{F}$ such that g induces Φ . One can similarly define pointed free splittings (Γ, G, \star) and based relative expansions $g: (\Gamma, G, \star) \to (\Gamma, G, \star)$. **Example 2.11.** Suppose $\mathbb{F} = \langle a, b \rangle$ and $\Phi' \colon \mathbb{F} \to \mathbb{F}$ maps $a \mapsto b^{-2}a^{-1}b^2$ and $b \mapsto b^{-1}ab$. Let the endomorphism Φ , the marked barbell B, and the relative expansion h be the same as in Example 2.8. The endomorphisms Φ, Φ' represent the same outer class ϕ . Consider the unit interval [0, 1] as a graph with one edge. We construct a pointed core graph (B', 0) as follows: subdivide the separating edge $m \subset B$ at a point $p \in m$, then attach the unit interval by identifying $1 \in [0, 1]$ with $p \in B$ (see Fig. 3). The deformation retraction $B' \to B$ induces a marking $\pi_1(B', 0) \cong \mathbb{F}$. After applying a homotopy supported on m (if necessary), the relative expansion h extends to a based relative expansion h' on (B', G, 0) that represents Φ' . As $0 \notin G$, the stable image of Φ' is trivial; meanwhile, the endomorphism Φ has the nontrivial stable image $\langle a \rangle$.



Figure 3: A marked pointed core graph and a based relative expansion.

We now show that relative expansions always extend to based relative expansions:

Theorem 2.7. If the endomorphism Φ is not surjective, then it is represented by a based relative expansion $g: (\Gamma, G, \star) \to (\Gamma, G, \star)$ whose peripheral restriction is an almost homotopy equivalence.

The next proof is almost identical to our proof of Theorem 2.4; the minor difference is that we keep track of the basepoint. To avoid repetition, we only sketch the proof and leave it to the reader to fill in the details.

Sketch of proof. Let $g': (\Gamma', G) \to (\Gamma', G)$ be the canonical relative expansion for ϕ . Construct Γ_0 by identifying $1 \in [0, 1]$ with an arbitrary vertex in Γ' and choose the basepoint $\star_0 = 0 \in [0, 1] \subset \Gamma_0$. We can now naively extend the marking $\pi_1(\Gamma') \cong \mathbb{F}$ to $\pi_1(\Gamma_0, \star_0) \cong \mathbb{F}$ and the relative expansion g' to a based relative cellular map $g_0: (\Gamma_0, G, \star_0) \to (\Gamma_0, G, \star_0)$ that represents Φ .

Let $S(\Phi^n)_{\star} \to (\Gamma_0, \star_0)$ be the Stallings based map for $\Phi^n(\mathbb{F})$ over (Γ_0, \star_0) . Analogous to our proof of Theorem 2.4, construct a Lipschitz based natural map $g_N \colon S(\Phi^N)_{\star} \to S(\Phi^N)_{\star}$ with cancellation constant $C(g_N) \leq 2C$ and a g_N -invariant relatively short proper subgraph $G_N \subset S(\Phi^N)$. As before, relatively collapsing G_N produces the required based relative expansion $g \colon (\Gamma, G, \star) \to (\Gamma, G, \star)$. In essence, this proof shows that we could have carefully extended:

- the core graph Γ' to a pointed core graph (Γ, \star) with $\operatorname{core}(\Gamma) = \Gamma'$;
- the marking $\pi_1(\Gamma') \cong \mathbb{F}$ to $\pi_1(\Gamma, \star) \cong \mathbb{F}$; and
- the relative expansion g' to a based relative expansion g on (Γ, G, \star) .

By bounded cancellation again, there is essentially a unique based relative expansion representing Φ whose peripheral restriction is an almost homotopy equivalence; we call this the canonical based relative expansion for Φ . Note that the proofs of Theorems 2.4, 2.5, and $\overline{2.7}$ are constructive:

 \square

Corollary 2.8. There is an algorithm that finds the canonical based relative expansion when the endomorphism Φ is not surjective.

Corollary 2.9. There is an algorithm that finds a basis for the stable image $\Phi^{\infty}(\mathbb{F})$.

Algorithm. If Φ is an automorphism, then $\Phi^{\infty}(\mathbb{F}) = \mathbb{F}$; return any basis of \mathbb{F} . Otherwise, find the canonical based relative expansion on (Γ, G, \star) for Φ (Corollary 2.8). If the basepoint \star is not in G, then $\Phi^{\infty}(\mathbb{F})$ is trivial; return the empty set. Otherwise, $\Phi^{\infty}(\mathbb{F})$ is identified with $\pi_1(G, \star)$ via the marking $\pi_1(\Gamma, \star) \cong \mathbb{F}$. In particular, $\Phi^{\infty}(\mathbb{F})$ is a free factor of \mathbb{F} ; return any basis for $\pi_1(G, \star) \leq \pi_1(\Gamma, \star)$.

The first step of the previous algorithm is to check whether Φ is an automorphism. We encourage the reader to use Stallings' paper [Sta83] to give a procedure that decides whether an endomorphism of \mathbb{F} is injective/surjective.

Exercise 2.12. There is an algorithm that decides whether an endomorphism of \mathbb{F} is injective.

Exercise 2.13. There is an algorithm that decides whether an endomorphism of \mathbb{F} is surjective.

3 Pullbacks in free groups

In this section, we will introduce a second condition on ϕ needed for hyperbolization.

3.1 Pullbacks of immersions

Let $A \leq \mathbb{F}$ be a finitely generated nontrivial subgroup; its <u>reduced rank</u> is defined as $\operatorname{rr}(A) = \operatorname{rank}(A) - 1$. Hanna Neumann proved that a nontrivial intersection of any two finitely generated subgroups $A, B \leq \mathbb{F}$ satisfies $\operatorname{rr}(A \cap B) \leq 2\operatorname{rr}(A)\operatorname{rr}(B)$ and conjectured that the "2" in the upper bound could be removed [Neu57].

Let $g_i \colon \Gamma_i \to \Gamma$ be immersions of core graphs for i = 1, 2. In the product $\Gamma_1 \times \Gamma_2$, define the unreduced pullback as the subspace

$$\Gamma_{12} = \{ (x_1, x_2) \in \Gamma_1 \times \Gamma_2 : g_1(x_1) = g_2(x_2) \};$$

this is homeomorphic to a graph, and it inherits a locally injective map $g: \Gamma_{12} \to \Gamma$ given by $g: (x_1, x_2) \mapsto g_i(x_i)$. We endow Γ_{12} with a CW-structure so that g is an immersion that maps edges to edges. Stallings gives a general construction of the unreduced pullback of two cellular maps that map edges to edges [Sta83, §1.3]. The core of the graph Γ_{12} is denoted $\Gamma_1 \times_{\Gamma} \Gamma_2$, and the (reduced) pullback of (g_1, g_2) is the restriction $s: \Gamma_1 \times_{\Gamma} \Gamma_2 \to \Gamma$ of the immersion g to the core.

Example 3.1. Let \mathbb{S}^1 be the rose with one edge and $g_i \colon \Gamma_i \to \mathbb{S}^1$ the cover with d_i sheets for i = 1, 2. The pullback of (g_1, g_2) is the disjoint union of $gcd(d_1, d_2)$ connected covers of \mathbb{S}^1 , each with $lcm(d_1, d_2)$ sheets.

Suppose $A_1, A_2 \leq \mathbb{F}$ are finitely generated nontrivial subgroups and Γ is a marked graph. Let $s_i: S_i \to \Gamma$ (i = 1, 2) be the Stallings map for $[A_i]$. Stallings observed that the pullback $s: S_1 \times_{\Gamma} S_2 \to \Gamma$ of (s_1, s_2) represents intersections [Sta83, Thm. 5.5]: if $A_1 \cap A_2$ is not trivial, then some component of s is the Stallings map for $[A_1 \cap A_2]$ over Γ . Stephen Gersten combined this with an Euler characteristic argument to give a topological proof of H. Neumann's inequality [Sta83, §7.7].

If a component of the pullback represents an intersection, what about the rest of the pullback? Let $A_1 \setminus \mathbb{F}/A_2$ be the set of double cosets for (A_1, A_2) , and denote by $\mathcal{O}(A_1, A_2)$ the subset consisting of double cosets A_1xA_2 such that $A_1 \cap xA_2x^{-1}$ is not trivial. There is a bijective correspondence between $\mathcal{O}(A_1, A_2)$ and the components of $s: S_1 \times_{\Gamma} S_2 \to \Gamma$, where each component is the Stallings map for $[A_1 \cap xA_2x^{-1}]$ with $A_1xA_2 \in \mathcal{O}(A_1, A_2)$. Extending Gersten's observation, Walter Neumann proved $-\chi(S_1 \times_{\Gamma} S_2) \leq 2\chi(S_1)\chi(S_2)$ and conjectured that the "2" could be removed [Neu90]; also known as the "strengthened Hanna Neumann conjecture", this was independently proven by Igor Mineyev [Min12] and Joel Friedman [Fri15], but it is not needed for our purposes.

3.2 Pullback stability

For $n \geq 1$, set $\mathcal{O}_n = \mathcal{O}(\Phi^n(\mathbb{F}), \Phi^n(\mathbb{F}))$. Pick a marked graph Γ , and let $s_n \colon S(\phi^n) \to \Gamma$ be the Stallings maps for $[\Phi^n(\mathbb{F})]$. The (iterated) pullbacks of ϕ are the pullbacks $\varphi_n \colon \Lambda_n \to \Gamma$ of (s_n, s_n) . By W. Neumann's inequality, $-\chi(\Lambda_n) \leq 2 \operatorname{rr}(\mathbb{F})^2$.

Example 3.2. Suppose $\mathbb{F} = \langle a, b \rangle$, the endomorphism $\Psi \colon \mathbb{F} \to \mathbb{F}$ maps $a \mapsto ab$ and $b \mapsto ba$, and $\psi = [\Psi]$ is its outer class. Let R be the marked rose from Example 2.4. Then each pullback $\Lambda_n = S(\psi^n) \times_R S(\psi^n) \to R$ of ψ is a disjoint union of immersions of a rose and two circles (see Fig. 4 for n = 1, 2). The core graph Λ_n is homeomorphic to the disjoint union of a rose and two circles for all $n \geq 1$; however, the volume $\operatorname{vol}(\Lambda_n)$ is $4 \cdot 2^n$.

Let $\widehat{\mathcal{O}}_n \subset \mathcal{O}_n$ consist of the double cosets $\Phi^n(\mathbb{F})x\Phi^n(\mathbb{F})$ such that $x \notin \Phi(\mathbb{F})$. Denote by $\widehat{\Lambda}_n \subset \Lambda_n$ the union of components of Λ_n corresponding to $\widehat{\mathcal{O}}_n$. As s_{n+1} factors through s_n , we have $\varphi_{n+1} = \varphi_n \circ \psi_{n+1}$ for some immersion $\psi_{n+1} \colon \Lambda_{n+1} \to \Lambda_n$. The immersion ψ_{n+1} restricts to an immersion $\widehat{\Lambda}_{n+1} \to \widehat{\Lambda}_n$; therefore, if $\widehat{\Lambda}_n$ is empty, then so is $\widehat{\Lambda}_{n+1}$; and if $\widehat{\Lambda}_n$ is



Figure 4: A marked rose and two pullbacks.

a disjoint union of circles, then so is $\widehat{\Lambda}_{n+1}$ (possibly empty). By W. Neumann's inequality, the components of $\widehat{\Lambda}_n$ are circles for $n \ge 2 \operatorname{rr}(\mathbb{F})^2$ (see [Mut21, Lem. 5.1.4]).

We say the <u>pullbacks of ϕ stabilize</u> if some $\widehat{\mathcal{O}}_n$ (equivalently, $\widehat{\Lambda}_n$) is empty — this is the second condition needed for hyperbolization!

Exercise 3.3. Let $\mathbb{F} = \langle a, b \rangle$ and Φ, Ψ be the endomorphisms from Example 2.4. The pullbacks of their outer classes ϕ, ψ stabilize.

We used canonical relative expansions to extend Brinkmann's characterization of hyperbolic outer automorphism (Theorem 1.4) to a characterization of hyperbolic outer endomorphisms (Proposition 2.6). We now use them again to give a (necessary and) sufficient condition for pullbacks to stabilize:

Proposition 3.1 (cf. [Mut21, Prop. 5.1.5]). If the pullbacks of the outer endomorphism ϕ do not stabilize, then some nontrivial elements $\Phi^m(x), x^d$ are conjugate in \mathbb{F} with $m, d \geq 2$.

Sketch of proof. Suppose $g: \Gamma \to \Gamma$ represents ϕ and the pullbacks of ϕ do not stabilize, i.e. $\widehat{\Lambda}_n \neq \emptyset$ for all $n \geq 1$. Then Φ is not surjective and $\widehat{\Lambda}_n$ has circle components for $n \geq 2 \operatorname{rr}(\mathbb{F})^2$; therefore, a component of $\widehat{\Lambda}_n$ is a pair of immersed loops (σ, σ') in Γ . Pick $r \geq 2 \operatorname{rr}(\mathbb{F})^2$, and consider the immersions $\psi_n \cdots \psi_{r+1} : \widehat{\Lambda}_n \to \widehat{\Lambda}_r$ for n > r. By the (infinitary) pigeonhole principle and axiom of countable choice, we can choose components (σ_n, σ'_n) of $\widehat{\Lambda}_n$ for $n \geq r$ such that (σ_r, σ'_r) is in the image of (σ_n, σ'_n) for n > r; in particular, the loop $g^{n-r}(\sigma_n)$ is homotopic to a power of σ_r for n > r.

Let g be an expansion. The following argument is due to Kapovich [Kap00, Prop. 3.7]. Note that $g^{n-r}(\sigma_n)$ wraps around σ_r for n > r. Since g is an expansion, we can pick $N \gg r$ so that $g^{N-r}(e)$ is longer than an arbitrarily large power of σ_r , say σ_r^{100} for all edges e in Γ . By the pigeonhole principle again, some edge e is contained in loops σ_n, σ_{n+m} , where $N \leq n < n + m \leq N + \operatorname{vol}(\Gamma)$. Since g is an immersion, $g^{n-r}(e)$ covers σ_r^{100} , and $g^m(\sigma_r^{100})$ wraps around σ_r . If the arbitrarily large power σ_r^{100} was chosen properly in terms of σ_r , then a final pigeonhole principle argument implies $g^m(\sigma_r)$ is homotopic to a power σ_r^d with $|d| \geq 2$. Thus σ_r represents a nontrivial conjugacy class [x] in \mathbb{F} such that $[\Phi^m(x)] = [x^d]$. Double m and square d to ensure $m, d \geq 2$. In general, let g be the canonical relative expansion. As the peripheral restriction of g is an almost homotopy equivalence, the immersed loops σ_n $(n \ge r)$ are not peripheral. So they contain nonperipheral edges and a relative version of Kapovich's argument applies. \Box

One can show directly that the converse holds, and we leave that as an elementary (but challenging) exercise; alternatively, the converse is Theorem $4.7(5\Rightarrow 4)$ below.

Exercise 3.4. If some nontrivial elements $\Phi^m(x), x^d$ are conjugate in \mathbb{F} with $m, d \ge 2$, then the pullbacks of the outer endomorphism ϕ do not stabilize.

The sketched proof of Proposition 3.1 invokes the axiom of countable choice and an infinitary pigeonhole principle (if infinitely many pigeons are placed in finitely many holes, then some hole has infinitely many pigeons); however, a careful reverse engineering of the finitary pigeonhole principle (if nk pigeons are placed in k holes, then some hole has n pigeons) can strengthen the proposition and make the proof constructive.

Exercise 3.5. For a computable positive integer $N(\phi)$, $\widehat{\mathcal{O}}_{N(\phi)}$ is not empty if and only if some nontrivial elements $\Phi^m(x), x^d$ are conjugate in \mathbb{F} with $m, d \geq 2$.

As a corollary, the pullbacks of ϕ stabilize if and only if $\widehat{\mathcal{O}}_{N(\phi)}$ is empty.

4 Geometry of mapping tori

We are ready to discuss the geometry of $\mathbb{F}*_{\phi}$. For $\delta \geq 0$, a geodesic space is $\underline{\delta}$ -hyperbolic if any geodesic triangle is in the δ -neighborhood of any pair of its sides; e.g. the real hyperbolic *n*-spaces \mathbb{H}^n $(n \geq 2)$ are $\ln(1 + \sqrt{2})$ -hyperbolic. A finitely generated group is hyperbolic if its Cayley graph with respect to a finite generating set is δ -hyperbolic for some $\delta \geq 0$ [ABC⁺91, Def. 1.8, Prop. 2.1].

Now suppose G is the fundamental group of a finite 2-dimensional CW-complex X and the 1-skeleton $\widetilde{X}^{(1)}$ of the universal cover \widetilde{X} is not contractible. Let $\gamma \colon \mathbb{S}^1_{\gamma} \to \widetilde{X}^{(1)}$ be a simple loop in \widetilde{X} , i.e. an embedding of the circle; the domain \mathbb{S}^1_{γ} is a circle with a CWstructure such that γ maps edges to edges. The length $\ell(\gamma)$ of γ is the combinatorial length of its domain \mathbb{S}^1_{γ} . A (reduced) disk diagram for γ is a continuous extension $\delta \colon \mathbb{D}_{\delta} \to X$ of γ such that: \mathbb{D}_{δ} is the closed disk with some CW-structure and $\partial \mathbb{D}_{\delta} = \mathbb{S}^1_{\gamma}$; and δ homeomorphically maps *n*-cells to *n*-cells (n = 0, 1, 2). The area of γ , denoted area(γ), is the minimal number of 2-cells in \mathbb{D}_{δ} over all disk diagrams δ for γ . The Dehn function of G is the function $\mathbb{N} \to \mathbb{N}$ given by:

$$n \mapsto \max\{\operatorname{area}(\gamma) : \gamma \text{ is a simple loop in } X \text{ with } \ell(\gamma) \le n\}.$$

We say G has a linear (quadratic, exponential resp.) Dehn function if its Dehn function is bounded above and below by linear (quadratic, exponential resp.) functions. The group G has a linear Dehn function if and only if it is hyperbolic [ABC⁺91, Thm. 2.5, Prop. 2.10].

Here is the main observation needed to complete our hyperbolization theorem: hyperbolicity and pullback stability for ϕ imply hyperbolicity of $\mathbb{F}*_{\phi}$; more generally, the pullbacks of ϕ stabilize if and only if $\mathbb{F}*_{\phi}$ has a linear or quadratic Dehn function. These observations will follow from Bestvina–Feighn's combination theorem and its relativization.

4.1 Bestvina–Feighn's combination theorem

Our exposition on Bestvina–Feighn's combination theorem has been simplified and specialized to the mapping torus setting. The reader will find the most general version in [BF92].

Let Γ be a marked graph and $g: \Gamma \to \Gamma$ a cellular map representing ϕ . The (topological) <u>mapping torus</u> $M_g = \Gamma \times [0, 1]/\sim_g$ of g is the quotient by the identification $(p, 0) \sim_g (g(p), 1)$ for all $p \in \Gamma$; it has an induced marking $\pi_1(M_g) \cong \mathbb{F}_{*\phi}$. Consider the universal cover \widetilde{M}_g of M_g , and equivariantly collapse the $\widetilde{\Gamma}$ -cross-sections to get the <u>Bass–Serre tree</u> T_{ϕ} for $\mathbb{F}_{*\phi}$; the $\mathbb{F}_{*\phi}$ -action on T_{ϕ} has exactly one orbit of vertices and edges. Recall that the ascending HNN extension $\mathbb{F}_{*\phi}$ has the relative presentation

$$\mathbb{F}_{\phi} = \langle \mathbb{F}, t \mid txt^{-1} = \Phi(x), \ \forall x \in \mathbb{F} \rangle;$$

the element $t \in \mathbb{F}*_{\phi}$ from this presentation is known as the <u>stable letter</u>, and it acts freely on T_{ϕ} . Orient the axis in T_{ϕ} for t so that t acts positively on it; the orientation equivariantly extends to all edges of T_{ϕ} to give the <u>t-orientation</u> on T_{ϕ} (see Fig. 5, left). Every vertex of T_{ϕ} has exactly one incoming edge with respect to the *t*-orientation; when Φ is surjective, T_{ϕ} is a line. The tree T_{ϕ} has a unique vertex/basepoint \star with stabilizer $\mathbb{F} \leq \mathbb{F}*_{\phi}$; this basepoint is on the axis for t. See [Ser77, §I.5.3] for another definition of T_{ϕ} .



Figure 5: A Bass–Serre tree and two conjugacies of length 4: flaring unidirectional and nonflaring strictly bidirectional. The class $[x_{\rho}]$ is represented by some map $\mathbb{S}^1 \times [-2, 2] \to M_q$.

For $L \geq 1$, a path-element pair in the system (\mathbb{F}, ϕ) of length 2L is a pair (ρ, x_{ρ}) of an immersed edge-path ρ in T_{ϕ} and an element $x_{\rho} \in \mathbb{F}*_{\phi}$ that fixes ρ (pointwise).

Elements $x \in \mathbb{F}_{*\phi}$ act on path-element pairs by $x \cdot (\rho, x_{\rho}) = (x \cdot \rho, xx_{\rho}x^{-1})$. A <u>conjugacy</u> $[\rho, x_{\rho}]$ is a $\mathbb{F}_{*\phi}$ -orbit of path-element pair (ρ, x_{ρ}) . The conjugacy $[\rho, x_{\rho}]$ is <u>unidirectional</u> if ρ is monotone (with respect to the *t*-orientation on T_{ϕ}); it is <u>strictly bidirectional</u> if ρ decomposes into two monotone pieces of length L (see Fig. 5, center/right). When Φ is surjective, all conjugacies in (\mathbb{F}, ϕ) are unidirectional as T_{ϕ} is a line.

Fix a conjugacy $[\rho, x_{\rho}]$ in the system (\mathbb{F}, ϕ) of length 2L. Let $(v_i)_{i=-L}^{L}$ be the consecutive vertices of ρ . For $-L \leq i \leq L$, choose an element $y_i \in \mathbb{F}_{*\phi}$ such that $v_i = y_i \cdot \star$. Then $x_i = y_i^{-1} x_{\rho} y_i \in \mathbb{F}$, and let $||x_i||$ be the Γ -length of the conjugacy class of x_i in \mathbb{F} . The conjugacy $[\rho, x_{\rho}]$ flares if $2 \cdot ||x_0|| \leq \max(||x_{-L}||, ||x_L||)$. The system (\mathbb{F}, ϕ) has the conjugacies flare property if, for some $L \geq 1$, all conjugacies in (\mathbb{F}, ϕ) of length 2L flare.

Example 4.1. Suppose $\mathbb{F} = \mathbb{Z}$ and the endomorphism $\hat{d}: \mathbb{Z} \to \mathbb{Z}$ is multiplication by $d \neq 0$. We abuse notation and refer the outer class $[\hat{d}]$ as \hat{d} too (since \mathbb{Z} is abelian and the outer class is a singleton). Note that $\mathbb{Z}*_{\hat{d}} = BS(1, d)$. Let $s \in BS(1, d)$ be the stable letter and $T_{\hat{d}}$ the Bass–Serre tree for BS(1, d) with basepoint $\star \in T_{\hat{d}}$. To describe a conjugacy in the system (\mathbb{Z}, \hat{d}) of length 2L, it is enough to give an immersed edge-path ρ in $T_{\hat{d}}$ of length 2L with middle vertex $v_0 = \star$ and an element $n_0 \in \mathbb{Z}$ that fixes ρ .

The following determines a unidirectional conjugacy in (\mathbb{Z}, d) of length 4: let ρ be the monotone edge-path with vertices $v_i = s^i \cdot \star$ for $-2 \leq i \leq 2$ and $n_0 = 3d^2$. As $sns^{-1} = nd$ in BS(1, d) for all $n \in \mathbb{Z}$, we have $n_{-2} = 3d^4$, $n_{-1} = 3d^3$, $n_0 = 3d^2$, $n_1 = 3d$, and $n_2 = 3$; this conjugacy flares if and only if $|d| \geq 2$: $||n_{-2}|| = 3d^4 = d^2 \cdot ||n_0||$. In fact, all unidirectional conjugacies in (\mathbb{Z}, \hat{d}) flare if and only if $|d| \geq 2$ since \hat{d} is represented by an immersion on a rose with one petal that sends the petal to an edge-path of length |d|; moreover, when $|d| \geq 2$, the unidirectional conjugacies have a unique direction of flaring: the reversal of the direction induced by the s-orientation (see Fig. 5, center).

Now suppose $|d| \geq 2$. The following determines a strictly bidirectional conjugacy in (\mathbb{Z}, \hat{d}) of length 4: let ρ' be the edge-path with vertices $v'_0 = \star, v'_i = s^i \cdot \star$, and $v'_{-i} = 1 \cdot v'_i$ for $0 \leq i \leq 2$ and $n'_0 = 3d^2$. Then $n'_{\pm 1} = 3d$ and $n'_{\pm 2} = 3$. This conjugacy does not flare, but its monotone pieces are conjugacies in (\mathbb{Z}, \hat{d}) of length 2 that flare (see Fig. 5, right). This generalizes to a family of nonflaring strictly bidirectional conjugacies in (\mathbb{Z}, \hat{d}) of length 2L for all $L \geq 1$; therefore, the system (\mathbb{Z}, \hat{d}) does not have the conjugacies flare property!

The system (\mathbb{F}, ϕ) has the <u>unidirectional conjugacies flare</u> property if, for some $L \geq 1$, all unidirectional conjugacies in (\mathbb{F}, ϕ) of length 2L flare. Let $[\rho, x_{\rho}]$ be a unidirectional conjugacy in (\mathbb{F}, ϕ) of length 2L, and consider the edge-path ρ with the *t*-orientation. The conjugacy classes (in \mathbb{F}) of the corresponding elements $x_i \in \mathbb{F}$ satisfy $[\Phi(x_i)] = [x_{i-1}]$ for $-L < i \leq L$. By unpacking the definitions of hyperbolicity and unidirectional conjugacy flaring, the reader should deduce that they are equivalent.

Exercise 4.2. The outer endomorphism ϕ is hyperbolic if and only if the system (\mathbb{F}, ϕ) has the unidirectional conjugacies flare property.

Similarly, unpacking the definition of strictly bidirectional conjugacies in (\mathbb{F}, ϕ) of length 2L gives a correspondence between them and elements of $\widehat{\mathcal{O}}_L$, i.e. the double cosets $\Phi^L(\mathbb{F})x\Phi^L(\mathbb{F})$ (in \mathbb{F}) such that $x \notin \mathbb{F}$ and $\Phi^L(\mathbb{F}) \cap x\Phi^L(\mathbb{F})x^{-1}$ is not trivial.

Exercise 4.3 (cf. [Mut21, Lem. 5.2.3]). The pullbacks of the outer endomorphism ϕ stabilize if and only if there is a constant $L \geq 1$ such that all strictly bidirectional conjugacies in the system (\mathbb{F}, ϕ) have length at most 2L.

Proposition 4.1. The outer endomorphism ϕ is hyperbolic and its pullbacks stabilize if and only if the system (\mathbb{F}, ϕ) has the conjugacies flare property.

Sketch of proof. Any conjugacy is a concatenation of a unidirectional conjugacy with a strictly bidirectional conjugacy. Hyperbolicity of ϕ is equivalent to the flaring of unidirectional conjugacies (Exercise 4.2). Pullback stability for ϕ is equivalent to a uniform bound on the length of strictly bidirectional conjugacies (Exercise 4.3). Together, they imply conjugacies flare in (\mathbb{F}, ϕ) — sufficiently long conjugacies will be almost unidirectional and, hence, will flare: see the second paragraph of the proof of Proposition 4.4 below.

Conversely, if the pullbacks of ϕ do not stabilize, then, by the proof of Proposition 3.1, there are integers $m, d_{\pm} \geq 2$ and a strictly birectional conjugacy $([\rho], [x_{\rho}])$ with length $2r \gg 2m$ such that $[\Phi^m(x_{\pm r})] = [x_{\pm r}^{d_{\pm}}]$ in \mathbb{F} . In particular, the monotone pieces of the conjugacy are two flaring unidirectional conjugacies joined at the flaring end; this strictly bidirectional conjugacy does not flare (see Fig. 5, right)! As $r \gg m$ is arbitrary, the system (\mathbb{F}, ϕ) does not have the conjugacies flare property. \Box

Theorem 4.2 (combination [BF92]). If the system (\mathbb{F}, ϕ) has the conjugacies flare property, then the mapping torus $\mathbb{F}_{*_{\phi}}$ is hyperbolic.

Technically, Bestvina–Feighn assume "annuli flare" rather than conjugacies flare. The annuli flare property is more difficult to define, yet not more enlightening. We refer the interested reader to the proofs of [Mut21, Lem. 5.2.3-5.2.4, Thm. 5.3.7] for an idea on how to upgrade conjugacy flaring to annuli flaring (via Proposition 4.1). Alternatively, the next subsection has another proof of the next theorem's main implication $(3 \Rightarrow 1)$ that is independent of Propositions 2.6 and 4.1. We are now ready to prove hyperbolization:

Theorem 4.3 (cf. [Mut21, Thm. 5.2.7]). The following are equivalent:

- 1. the mapping torus \mathbb{F}_{ϕ} is hyperbolic;
- 2. \mathbb{F}_{ϕ} has no BS(1, d)-subgroup for all $d \geq 1$;
- 3. for all $n, d \ge 1$ and nontrivial $x \in \mathbb{F}$, the elements $\Phi^n(x), x^d$ are not conjugate in \mathbb{F} ; and
- 4. the outer endomorphism ϕ is hyperbolic and its pullbacks stabilize.

Proof. The implications $(1 \Rightarrow 2 \Rightarrow 3)$ follow from the standard Exercises 4.4 and 4.5; $(3 \Rightarrow 4)$ is Propositions 2.6 and 3.1; and $(4 \Rightarrow 1)$ is Proposition 4.1 and Theorem 4.2.

Exercise 4.4. Hyperbolic groups have no BS(1, d)-subgroups for all $d \ge 1$.

Hint. \mathbb{Z} -subgroups of hyperbolic groups are undistorted [ABC⁺91, Prop. 3.2]; show that this rules out BS(1,d) for $d \geq 2$. Centralizers of \mathbb{Z} -subgroups in hyperbolic groups are virtually- \mathbb{Z} [ABC⁺91, Prop. 3.5]; this rules out BS(1,1).

Exercise 4.5. If $\Phi^n(x) = g^{-1}x^dg$ in \mathbb{F} for some $n, d \ge 1$, element $g \in \mathbb{F}$, and nontrivial $x \in \mathbb{F}$, then the subgroup $\langle x, gt^n \rangle \le \mathbb{F} *_{\phi}$ is naturally isomorphic to BS(1, d).

Hint. First show that there is a natural homomorphism $BS(1,d) \to \mathbb{F}_{\phi}$ onto $\langle x, gt^n \rangle$. Then use the normal forms of elements in BS(1,d) and \mathbb{F}_{ϕ} to prove that this homomorphism is injective; normal forms are called "reduced words" in [Ser77, §I.5].

Example 4.6. Suppose $\mathbb{F} = \langle a, b \rangle$ and $\Psi \colon \mathbb{F} \to \mathbb{F}$ maps $a \mapsto ab$ and $b \mapsto ba$. An expansion represents the outer class $\psi = [\Psi]$ (Example 2.4), and ψ is hyperbolic (Exercise 2.9). Since the pullbacks of ψ stabilize (Exercise 3.3), the mapping torus

$$\mathbb{F}*_{\psi} = \langle a, b, t | tat^{-1} = ab, tbt^{-1} = ba \rangle$$

is hyperbolic (Theorem 4.3: $4 \Rightarrow 1$); this mapping torus is also known as Sapir's group.

4.2 Mj–Reeves' relative combination theorem

Beyond hyperbolicity, we can still study the Dehn function of $\mathbb{F}*_{\phi}$. To this end, we will sketch a relative version of Theorem 4.3 when Φ is not surjective — this is a new result!

Suppose $g: (\Gamma, G) \to (\Gamma, G)$ is the canonical relative expansion for ϕ . As g is a relative expansion, the outer endomorphism ϕ is positively hyperbolic rel. G: there is a constant $L \geq 1$ such that $2 \cdot \|x\|_G \leq \|\Phi^L(x)\|_G$ for all $x \in \mathbb{F}$, where $\|x\|_G$ is the relative length of the immersed (or constant) loop in Γ representing [x].

The free splitting (Γ, G) lets us represent the sequence $([x_i])_{i=-L}^L$ for a conjugacy in (\mathbb{F}, ϕ) as a sequence $(\sigma_i)_{i=-L}^L$ of immersed loops in Γ . A conjugacy in (\mathbb{F}, ϕ) flares rel. G if $2 \cdot \|\sigma_0\|_G \leq \max(\|\sigma_{-L}\|_G, \|\sigma_L\|_G)$. The system (\mathbb{F}, ϕ) has the conjugacies flare rel. G property if, for some $L \geq 1$, all conjugacies in (\mathbb{F}, ϕ) of length 2L flare rel. G.

Proposition 4.4. Suppose the endomorphism Φ is not surjective, and $g: (\Gamma, G) \to (\Gamma, G)$ is the canonical relative expansion for its outer class ϕ . The system (\mathbb{F}, ϕ) has the conjugacies flare rel. G property if and only if the pullbacks of ϕ stabilize.

Proof. Since the outer endomorphism ϕ is positively hyperbolic rel. G, the system (\mathbb{F}, ϕ) has the unidirectional conjugacies flare rel. G property with a unique direction for flaring given by the reversal of the *t*-orientation; therefore, a long strictly bidirectional conjugacy is two

flaring unidirectional conjugacies joined at the flaring ends (see Fig. 5, right). This means long strictly bidirectional conjugacies cannot flare. So the conjugacy flare rel. G property for (\mathbb{F}, ϕ) implies a uniform bound on the length of strictly bidirectional conjugacies. In light of Exercise 4.3, it remains to show the converse holds too: a uniform bound on the length of strictly bidirectional conjugacies implies (\mathbb{F}, ϕ) has the conjugacies flare rel. Gproperty.

Suppose all strictly bidirectional conjugacies in (\mathbb{F}, ϕ) have length at most $2L_0$, the outer endomorphism ϕ has $L_1 \geq 1$ as its length of positive hyperbolicity rel. G, and the relative expansion g is K-Lipschitz ($K \geq 1$). For $n \geq 1$, a conjugacy $[\rho, x_\rho]$ in (\mathbb{F}, ϕ) of length $2nL_0L_1$ is a concatenation of monotone pieces, one of length $M \leq L_0$. Up to an orientation of ρ , we have $[\Phi^{nL_0L_1}(x_0)] = [\Phi^{2M}(x_{-nL_0L_1})]$ (see Fig. 6). Since



Figure 6: A conjugacy in (\mathbb{F}, ϕ) of length $2nL_0L_1$ whose strictly bidirectional part has length $2M \leq 2L_0$.

$$\begin{split} \|\Phi^{nL_0L_1}(x_0)\|_G &\geq 2^{nL_0} \|x_0\|_G \text{ and } \|\Phi^{2M}(x_{-nL_0L_1})\|_G \leq K^{2M} \|x_{-nL_0L_1}\|_G, \text{ we get} \\ \|x_{-nL_0L_1}\|_G &\geq \frac{2^{nL_0}}{K^{2M}} \|x_0\|_G \geq \frac{2^{nL_0}}{K^{2L_0}} \|x_0\|_G, \end{split}$$

and the conjugacy flares as long as $2^{nL_0} \ge K^{2L_0}$ — this can be arranged by choosing $n \gg 1$. In other words, the system (\mathbb{F}, ϕ) has the conjugacies flare rel. *G* property.

Let the expansion $\widehat{g}: \widehat{\Gamma} \to \widehat{\Gamma}$ be the Φ -equivariant "lift" of g to the relative universal cover of $(\Gamma, G) - \underline{\Phi}$ -equivariant means $\widehat{g}(x \cdot p) = \Phi(x) \cdot \widehat{g}(p)$ for all $x \in \mathbb{F}$ and $p \in \widehat{\Gamma}$. The <u>G</u>-relative (universal) cover \widehat{M}_g of M_g is constructed by equivariantly collapsing the \widetilde{G} -cross-sections in the universal cover \widetilde{M}_g of M_g ; moreover, equivariantly collapsing the $\widehat{\Gamma}$ -cross-sections in \widehat{M}_g produces the Bass–Serre tree T_{ϕ} . For each vertex $v \in T_{\phi}$, denote the corresponding $\widehat{\Gamma}$ -cross-section in \widehat{M}_g by $\widehat{\Gamma}_v$. For each half-open edge $(v, w] \subset T_{\phi}$ (with the *t*-orientation), its preimage in \widehat{M}_g is homeomorphic to the product $\widehat{\Gamma}_w \times (v, w]$, and the preimage's attaching map to $\widehat{\Gamma}_v$ is the expansion $\widehat{g}_{wv}: \widehat{\Gamma}_w \cong \widehat{\Gamma}_w \times \{v\} \to \widehat{\Gamma}_v$, which is identified with some lift $\widehat{\Gamma} \to \widehat{\Gamma}$ of g under the identifications $\widehat{\Gamma}_w \cong \widehat{\Gamma} \cong \widehat{\Gamma}_v$.

For $L \geq 1$, a hallway in the *G*-relative cover \widehat{M}_g of length 2L is a pair $(\rho, (\rho_i)_{i=-L}^L)$, where $\rho = (v_i)_{i=-L}^L$ is an immersed edge-path in T_{ϕ} of length 2L and ρ_i is an immersed edge-path in $\widehat{\Gamma}_{v_i} \subset \widehat{M}_g$ for $-L \leq i \leq L$. The hallway is <u>unidirectional</u> if ρ is monotone (with respect to the *t*-orientation on T_{ϕ}); it is <u>strictly bidirectional</u> if ρ decomposes into two monotone pieces of length L.

The girth of a hallway $(\rho, (\rho_i)_{i=-L}^L)$ is the combinatorial length $|\rho_0|$ of the edge-path ρ_0 . The hallway flares if $2 \cdot |\rho_0| \leq \max(|\rho_{-L}|, |\rho_L|)$. The hallway is <u>r-thin</u> $(r \geq 0)$ if the combinatorial distance between the initial (terminal resp.) vertices of ρ_i and $\hat{g}_{j,i}(\rho_j)$ is at most r whenever |i - j| = 1 and (v_i, v_j) is the t-orientation between the vertices — here, $\hat{g}_{j,i}: \hat{\Gamma}_{v_j} \to \hat{\Gamma}_{v_i}$ are the maps governing the attaching in \hat{M}_g . The hallway is <u> \tilde{G} -bounded</u> if it is 0-thin and the initial and terminal vertices of ρ_i correspond to lifts of G for $-L \leq i \leq L$.

Example 4.7. Suppose $\mathbb{F} = \langle a, b \rangle$ and $\Phi \colon \mathbb{F} \to \mathbb{F}$ maps $a \mapsto a^{-1}$ and $b \mapsto bab^{-1}$. The relative expansion $h \colon (B, G) \to (B, G)$ given in Example 2.8 is the canonical relative expansion for the outer class $\phi = [\Phi]$. Let $u \in B$ be the vertex of the marked graph fixed by h. Since h induces Φ on $\pi_1(B, u) \cong \mathbb{F}$, the Φ -equivariant lift $\tilde{h} \colon \tilde{B} \to \tilde{B}$ to the universal cover will fix a lift \tilde{u} of u. Choose the (oriented) lifts $\tilde{\ell}, \tilde{m}\tilde{r}$ that start at \tilde{u} of the edge-paths ℓ, mr respectively; so $\tilde{h}(\tilde{m}) = \tilde{m}\tilde{r}(b \cdot \tilde{m})$ (see Fig. 7). The relative universal cover \hat{B} of (B, G) has only one orbit of edges, and \hat{m} is the image of \tilde{m} under the equivariant collapse map $\tilde{B} \to \hat{B}$. Let $\hat{h} \colon \hat{B} \to \hat{B}$ be the induced Φ -equivariant expansion; so $\hat{h}(\tilde{m}) = \hat{m}(b \cdot \tilde{m})$.

We now give a unidirectional hallway in the *G*-relative cover \widehat{M}_h of length 2. Let ρ be the immersed path with the vertices: $v_{-1} = t^{-1} \cdot \star$, $v_0 = \star$, and $v_1 = b^{-1}t \cdot \star$, where $\star \in T_{\phi}$ is the basepoint. Via $\widehat{B}_{v_i} \cong \widehat{B}$, we identify $\widehat{h}_{0,-1} \colon \widehat{B}_{v_0} \to \widehat{B}_{v_{-1}}$ with $\widehat{h} \colon \widehat{B} \to \widehat{B}$ and $\widehat{h}_{1,0} \colon \widehat{B}_{v_1} \to \widehat{B}_{v_0}$ with $b^{-1} \cdot \widehat{h} \colon \widehat{B} \to \widehat{B}$; similarly, we identify the immersed paths in \widehat{B}_{v_i} with immersed paths in \widehat{B} . Any three immersed paths $(\rho_i)_{i=-1}^1$ in \widehat{B} will determine a hallway in \widehat{M}_h . Let the first path be $\rho_1 = b \cdot \widehat{m}$, the second $\rho_0 = \Phi(b) \cdot (\widehat{m}(b \cdot \overline{m}))$, and the third $\rho_{-1} = \Phi^2(b) \cdot (\widehat{m}(b \cdot \overline{m})(\Phi(b)b \cdot \widehat{m})\overline{m})$. The hallway $(\rho, (\rho_i)_{i=-1}^1)$ flares and is \widetilde{G} -bounded.

The *G*-relative cover \widehat{M}_g has the \widetilde{G} -bounded hallways strictly flare property if, for some $L \geq 1$, all \widetilde{G} -bounded hallways in \widehat{M}_g of length 2L flare. The *G*-relative cover \widehat{M}_g has the hallways flare property if there are $L \geq 1$ and $H \colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that, for all $r \geq 0$, an r-thin hallway in \widehat{M}_g of length 2L flares if its girth is at least H(r).

To state Mj–Reeves' combination theorem, we introduce a peripheral structure on $\mathbb{F}*_{\phi}$. A $\underline{\phi}$ -peripheral free-by-cyclic subgroup of $\mathbb{F}*_{\phi}$ is a subgroup corresponding to a component of the mapping torus of the peripheral restriction g|G.

Theorem 4.5 (relative combination [MR08, Thm. 4.5]). Suppose the endomorphism Φ is not surjective, $g: (\Gamma, G) \to (\Gamma, G)$ is the canonical relative expansion for its outer class ϕ , and $\hat{g}: \hat{\Gamma} \to \hat{\Gamma}$ is the Φ -equivariant lift of g to the relative universal cover of (Γ, G) . If the G-relative cover \widehat{M}_g has the hallways flare and \widetilde{G} -bounded hallways strictly flare properties, then the mapping torus $\mathbb{F}*_{\phi}$ is (strongly) hyperbolic relative to the family of ϕ -peripheral free-by-cyclic subgroups.



Figure 7: The universal cover \hat{B} and the relative universal cover \hat{B} .

Besides generalizing hyperbolicity, the definition of relative hyperbolicity [Far98, Bow12] is not needed; we only need the fact that the Dehn function of a relatively hyperbolic group is the Dehn function of its peripheral subgroups [Far98, Thm. 3.8]. The following will allow us to apply Mj–Reeves' relative combination theorem.

Proposition 4.6. Suppose the endomorphism Φ is not surjective, $g: (\Gamma, G) \to (\Gamma, G)$ is the canonical relative expansion for ϕ , and $\widehat{g}: \widehat{\Gamma} \to \widehat{\Gamma}$ is the Φ -equivariant lift of g to the relative universal cover of (Γ, G) . If the system (\mathbb{F}, ϕ) has the conjugacies flare rel. Gproperty, then the G-relative cover \widehat{M}_g has the hallways flare and \widetilde{G} -bounded hallways strictly flare properties.

The reader should fill in the details of the following proof as the final exercise

Sketch of proof. Let $(\rho, (\rho_i)_{i=-L}^L)$ be a \widetilde{G} -bounded hallway in \widehat{M}_g . Each ρ_i concatenates with a translate of itself to form a fundamental domain of an axis of some element $x_i \in \mathbb{F}$. Thus, it determines a conjugacy in (\mathbb{F}, ϕ) that flares (rel. G) if and only if the original \widetilde{G} -bounded hallway flares; therefore, the conjugacies flare rel. G property for the system (\mathbb{F}, ϕ) implies the \widetilde{G} -bounded hallways strictly flare property for the G-relative cover \widehat{M}_g .

For the hallways flare property, we first argue that strictly bidirectional r-thin hallways with large enough (with respect to r) girth have uniformly bounded (independent of r) length. If $|\rho_0|$ is large enough, then $\hat{g}_{-1,0}(\rho_{-1})$, ρ_0 , $\hat{g}_{1,0}(\rho_1)$ coincide (modulo initial/terminal segments of length at most r). Thus the unreduced pullback of (g, g) contains a long enough (rel. G) immersed path that is not in the diagonal; since the unreduced pullback is fixed, it contains an immersed loop not in the diagonal. This determines a strictly bidirectional conjugacy in (\mathbb{F}, ϕ) of length 2. The same argument shows that if $|\rho_0|$ is large enough, then some strictly bidirectional conjugacy in (\mathbb{F}, ϕ) has the same length as the initial strictly bidirectional hallway; the conjugacy has a uniformly bounded length by conjugacy flaring.

The previous paragraph implies that sufficiently long r-thin hallways with large enough girth are almost unidirectional. As g is a relative expansion, one can verify the hallways flare property, starting with unidirectional hallways. A variation of this argument for annuli appears in [Mut21, Thm. 5.3.7].

We can now prove relative hyperbolization:

Theorem 4.7. If the endomorphism Φ is not surjective, then the following are equivalent:

- 1. the mapping torus \mathbb{F}_{ϕ} is hyperbolic relative to the family of ϕ -peripheral free-by-cyclic subgroups;
- 2. \mathbb{F}_{ϕ} has linear or quadratic Dehn function;
- 3. $\mathbb{F}_{*_{\phi}}$ has no BS(1, d)-subgroup for all $d \geq 2$;
- 4. for all $n, d \ge 2$ and nontrivial $x \in \mathbb{F}$, the elements $\Phi^n(x), x^d$ are not conjugate in \mathbb{F} ; and
- 5. the pullbacks of the outer endomorphism ϕ stabilize.

Proof. The implication $(3 \Rightarrow 4)$ follows from Exercise 4.5; $(4 \Rightarrow 5)$ is Proposition 3.1; and $(5 \Rightarrow 1)$ is Propositions 4.4 and 4.6, and Theorem 4.5.

 $(1 \Rightarrow 2)$: as free-by-cyclic groups have linear or quadratic Dehn functions [BG10], so does \mathbb{F}_{ϕ} [Far98, Thm. 3.8]. $(2 \Rightarrow 3)$: if \mathbb{F}_{ϕ} contains BS(1,d) for some $d \ge 2$, then it has an exponential Dehn function [Kap00, Cor. 5.7].

Corollary 4.8. The group $\mathbb{F}_{*_{\phi}}$ has a linear, quadratic, or exponential Dehn function. \Box

Second proof of Theorem 4.3: $3\Rightarrow 1$. By Theorem 1.3($3\Rightarrow 1$), assume Φ is not surjective. By Theorem 4.7($4\Rightarrow 1$), $\mathbb{F}*_{\phi}$ is hyperbolic relative to the family of ϕ -peripheral free-by-cyclic subgroups. These free-by-cyclic subgroups are hyperbolic by Theorem 1.3($3\Rightarrow 1$); therefore, $\mathbb{F}*_{\phi}$ is hyperbolic [Far98, Thm. 3.8].

Example 4.8. Suppose $\mathbb{F} = \langle a, b \rangle$ and $\Phi \colon \mathbb{F} \to \mathbb{F}$ maps $a \mapsto a^{-1}$ and $b \mapsto bab^{-1}$. The relative expansion $h \colon (B, G) \to (B, G)$ given in Example 2.8 is the canonical relative expansion for the outer class $\phi = [\Phi]$. The ϕ -peripheral free-by-cyclic subgroup is the Klein bottle group BS(1, -1). Since the pullbacks of ϕ stabilize (Exercise 3.3), the mapping torus

$$\mathbb{F}*_{\psi} = \langle a, b, t | tat^{-1} = a^{-1}, tbt^{-1} = bab^{-1} \rangle$$

is hyperbolic relative to the ϕ -peripheral Klein bottle subgroup $\langle a, t \rangle$ (Theorem 4.7:5 \Rightarrow 1).

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