Nielsen-Thurston Decomposition

ref: Casson-Bleiler, Automorphisms of Surfaces...

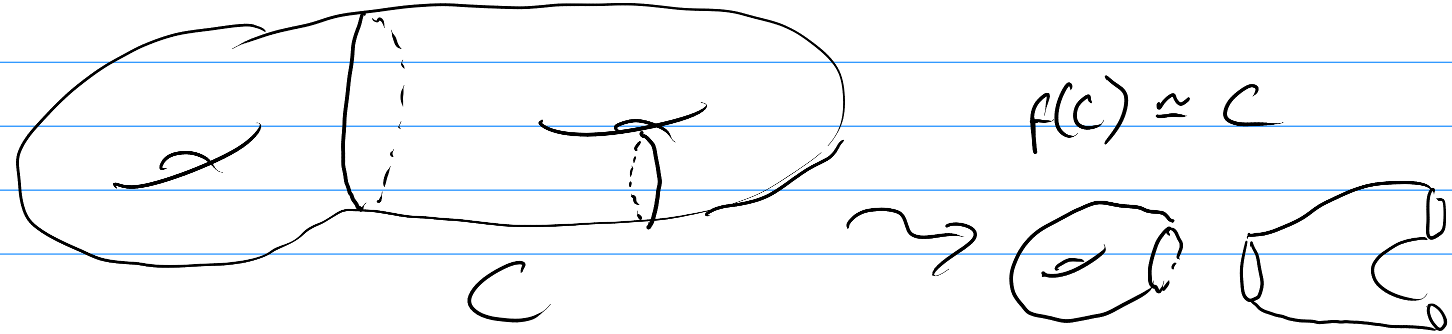
S compact surface w/ $\underbrace{\chi(S) < 0}_{\text{admits a hyp. structure}}$

$f: S \to S$ homeo.

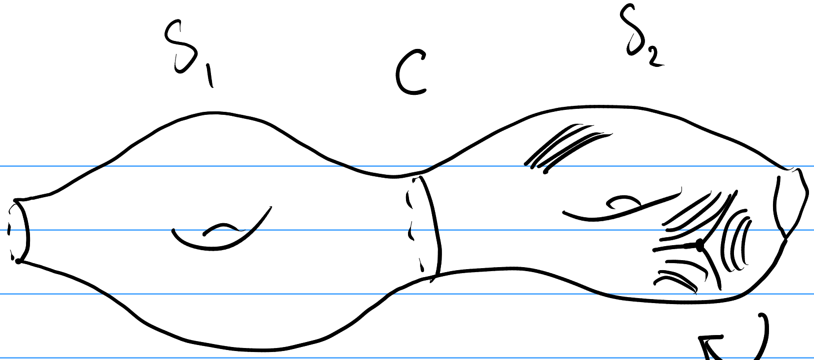
Goal: Give a canonical description of f up to isotopy.

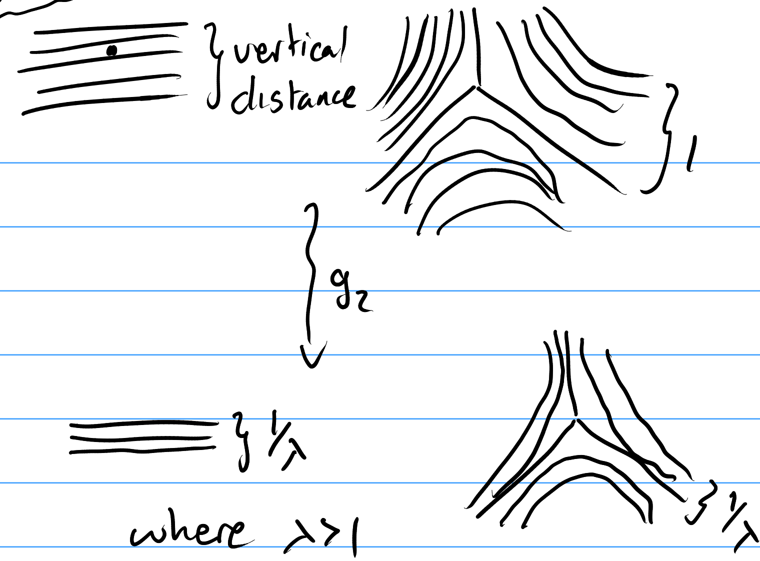
f has finite-order/periodic: $\exists\, n > 1$, f^n isotopic to Id_S

f is reducible if it fixes an essential multicurve



measured





Decomposition:

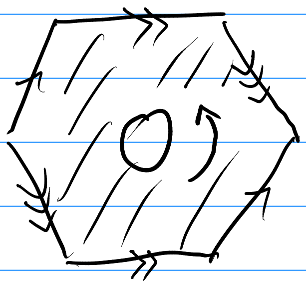
f is isotopic to $g : S \to S$

g_1 is reducible to finite-order

g_2 is **pseudo-Anosov**

$$\begin{pmatrix} \lambda & 0 \\ 0 & 1/\lambda \end{pmatrix}$$

where $\lambda > 1$



finite-order 6.

__Sketch__ : infinite order $\Rightarrow \exists$ simple loop with ∞-orbit (up to isotopy)

Pick a hyp. metric w/ geod. boundary

$\Rightarrow$ use iteration to find a fixed __geodesic__ __lamination__.

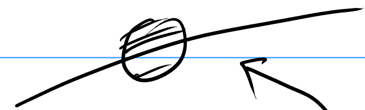
geod. lam. : $L \subseteq S$ closed and union of pairwise disjoint simple geodesic.

(involves a compactness argument, hyp. geom...)

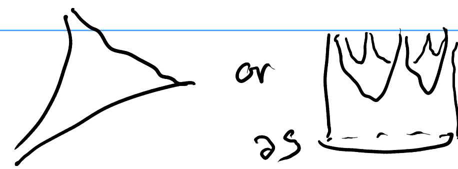
$f^n(\underbrace{\text{closed geod.}})$ $\rightsquigarrow$ fixed geod. lam. L

If f is reducible, take a maximal reducing curve C

$\rightsquigarrow$ may now assume f is irreducible.



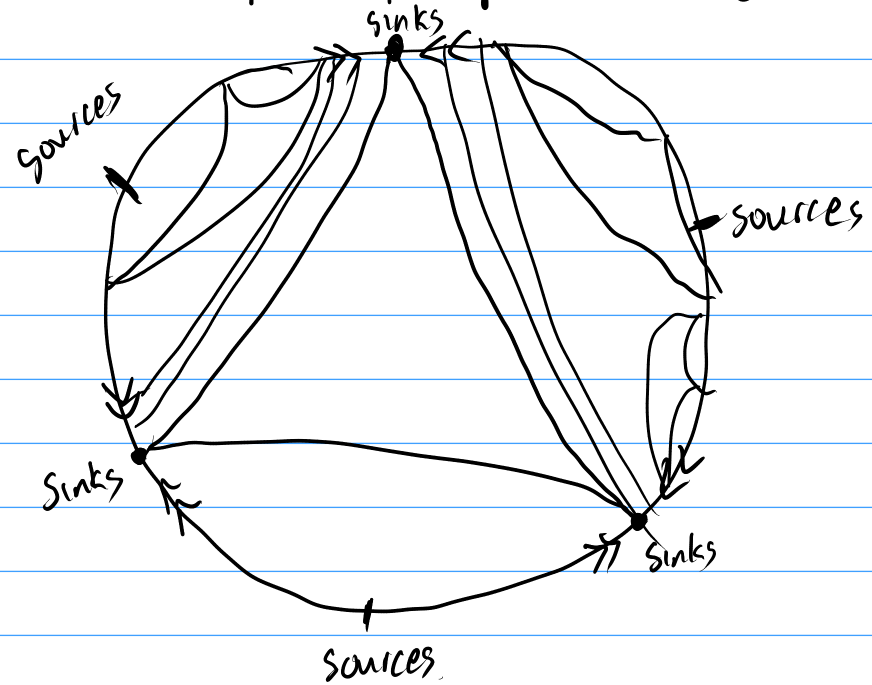
irred. + L fixed geod lamination $\Rightarrow$ L is filling & __perfect__.



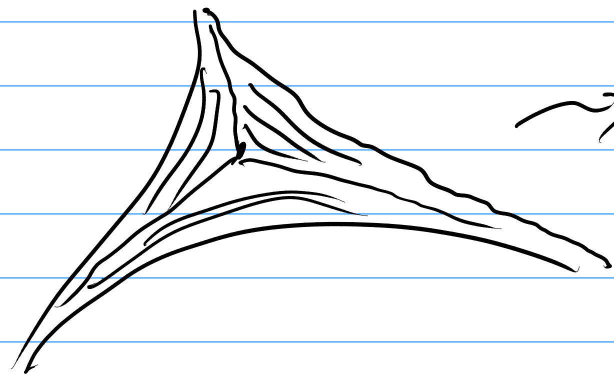
Need to show: L is independent of choices made so far.

Identify $\tilde{S}$ = convex subset of $\mathbb{H}^2$

$\tilde{L}$ = full lift of L to $\tilde{S}$.

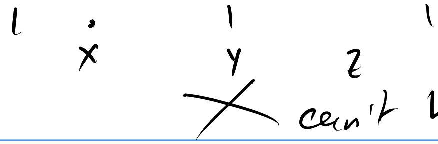


$\rightsquigarrow$ L is completely characterized by the dynamics on $\partial_\infty \tilde{S} \subseteq \widehat{\mathbb{H}^2} \setminus \mathbb{H}^2$



$\rightsquigarrow$ foliation of S.

[To find the measure: Markov partition to this into a linear algebra problem.]



can't happen.

Dual Pretree

Baby case: $P \subseteq [0,1]$ perfect: $x \sim y$ on $[0,1] \iff$ $x = y$ or $x, y \in$ closure of a conn. comp. of $[0,1] \setminus P$.

Then $\sim$ is an equiv. rel.

& $[0,1]/_{\sim} \underset{\text{order}}{\overset{\cancel{\text{homeo.}}}{\cong}} [0,1]$ ← not really a top. fact.

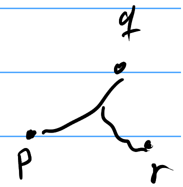
L perfect geod. lam. $\rightsquigarrow$ $\tilde{L}$ full lift to the universal cover $\tilde{S}$

$x \sim_L y$ on $\tilde{S} \iff x = y$ or $x, y \in$ closure of conn. comp. of $\tilde{S} \setminus \tilde{L}$.

Def/Prop: $T := \tilde{S}/_{\sim_L}$ is a natural real pretree:

$\left(T, [\cdot,\cdot] : T \times T \to \mathcal{P}(T)\right)$: $\forall p, q, r \in T$,
$(p,q) \mapsto S \subseteq T$

1. $[p,q] = [q,p] \supseteq \{p, q\}$
2. $[p,r] \subseteq [p,q] \cup [q,r]$



3. $\underbrace{r \in [p,q] \text{ \& } q \in [p,r] \Rightarrow q = r.}_{\text{interval have a linear order.}}$

pretree is real if $[p,q]$ is order-isomorphic to $[0,1]$
$(p \neq q)$

Limit Pretrees for Free Group Automorphisms

- Jean Pierre Mutanguha
(IAS)

Nielsen-Thurston Decomposition: S compact surface w/ $\chi(S) < 0$
$f: S \to S$ homeo.
$\phi: \pi_1(S) \to \pi_1(S)$ induced automorphism (outer automorphism)

$\exists$ {a real pretree T w/ convex metric} $\mathbb{R}$-tree T $\rightsquigarrow$ finitely many orbits of subtrees $T_1, \dots, T_n$

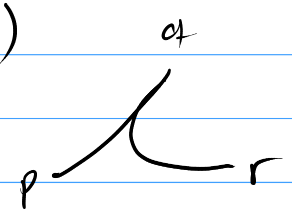
- $\pi_1(S) \curvearrowright T$ isometrically w/ trivial arc stabilizers
- $x \in \pi_1(S)$ is loxodromic $\Leftrightarrow$ x grows exponentially *
- $f: T \to T$ ϕ-equivariant expanding "affine homeomorphism".

& T is canonical: T' satisfying the same conclusion is equivariant pretree-isomorphic to T.

pretree is a set T
& function $[\cdot,\cdot] : T \times T \to \mathcal{P}(T)$
$\forall p, q, r \in T$

satisfying: ① $[p,q] = [q,p] \supseteq \{p,q\}$ (symmetric)

② $[p,r] \subseteq [p,q] \cup [q,r]$ (thin)



③ $r \in [p,q]$ & $q \in [p,r] \Rightarrow q = r$ (linearity)

a pretree is **real** if $\overrightarrow{[p,q]}$ is order-isomorphic to $[0,1]$ $(p \neq q)$

convex metric: $d(p,r) = d(p,q) + d(q,r)$ if $q \in [p,r]$.

Note: a real pretree w/ convex metrix $\Leftrightarrow$ $\mathbb{R}$-tree

$f : T \to T$ is **ϕ-equiv**: $f(x \cdot p) = \phi(x) \cdot f(p)$ $\forall\, x \in \pi_1(S),\ p \in T$

F f.g. free group

$\phi : F \to F$ automorphism

Y simplicial tree

$F \curvearrowright Y$ <u>geometric</u> :
- simplicial
- free
- cocompact ($F \backslash Y$ is compact)

$\| \cdot \|_Y : F \to \mathbb{R}_{\geq 0}$ length function

$$x \mapsto \inf_{p \in Y} d(p, x \cdot p) = \min_{p \in Y} d(p, x \cdot p)$$

$x \in F$, $\underbrace{\lambda(\phi, x)}_{\text{exp. growth factor of } x \text{ wrt } \phi} := \lim_{n \to \infty} \|\phi^n(x)\|_Y^{1/n} \in [1, \infty)$ $\leftsquigarrow$ independent of the geometric action $F \curvearrowright Y$.

$x \in F$ grows <u>exponentially</u> if $\lambda(\phi, x) > 1$,

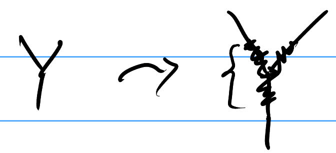
<u>polynomially</u> if $\lambda(\phi, x) = 1$.

F f.g. free group

$\phi : F \to F$ aut.

<u>Thm</u> (M.) $\exists$ real pretree T with:

- $F \curvearrowright T$ rigid/<u>non-nesting</u> w/ trivial arc stabilizer: $x \cdot [p,q] \subseteq [p,q] \Rightarrow x \cdot [p,q] = [p,q]$.
- $x \in F$ is loxodromic $\Leftrightarrow$ x grows exponentially
- $f : T \to T$ ϕ-equivariant "<u>F-expanding</u>" pretree-automorphism



<u>Missing</u>? Is T canonical? <u>idea</u>: • T should be independent of the choices made in the proof.

What polynomial dynamics?

In <u>Thm</u>, T is a point if ϕ has polynomial growth.

Sketch: If ϕ is poly. growing, then let T be a point. $\square$

Assume ϕ has exp. growth, i.e. $\exists\, x \in F$ that grows exponentially

Pick a geometric action $F \curvearrowright Y$

known by the mid-90s

exp. growing $\Rightarrow \exists\, \lambda > 1$ & $\|\cdot\| : F \to \mathbb{R}_{\geq 0}$ s.t. $\|\phi(\cdot)\| = \lambda \|\cdot\|$

("compactness argument")

~ $\|\cdot\|$ is not canonical

(Morgan-Shalen, Lustig):
- $\|\cdot\|$ is a length function for some F-action on an $\mathbb{R}$-tree $T^{(0)}$ by isometries
- $F \curvearrowright T^{(0)}$ has trivial arc stabilizers
- $\exists\, f^{(0)} : T^{(0)} \to T^{(0)}$ ϕ-equiv. expanding homothety $(\lambda > 1)$
- $x \in F$ loxodromic $\Rightarrow x$ grows exponentially

Problem: some elliptics might grow exponentially.

restrict ϕ to point stabilizers and repeat.

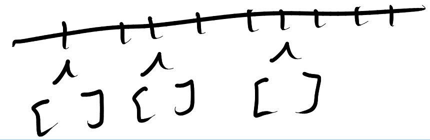
(Gaborian-Levitt)

$F \curvearrowright T^{(0)}$, $\underset{\substack{\parallel \\ \mathrm{Stab}(v),\ v \in T^{(0)}}}{G_1} \curvearrowright T^{(1)}$, ----, $\underset{\substack{\parallel \\ \mathrm{Stab}(w),\ w \in T^{(n-1)}}}{G_n} \curvearrowright T^{(n)}$

$x \in F$ is $T^{(0)}$-elliptic

$T^{(1)}$-elliptic

$\vdots$



- $x \in F$ is $T^{(n)}$-elliptic $\iff$ x grows polynomially

<u>New idea:</u> "blow-up" $T^{(0)}$ by equivariantly replacing $v \in T^{(0)}$ with $T^{(1)}$

- If $T^{(0)}$-directions at v are attached "appropriately", then $f^{(0)}$ & $f^{(1)}$ induce a ϕ-equiv. "expanding" pretree-automorphism.

- Since $f^{(1)}$ is exp. homothety; fixed point theorem $\Rightarrow$ $T^{(0)}$-directions can be attached appropriately to the metric completion $\overline{T}^{(1)}$.

$\square$

NTS: (for uniqueness)

$$\begin{array}{ccc} \partial F & \hookrightarrow & T \\ {\scriptstyle \partial\phi}\downarrow & & \downarrow{\scriptstyle f} \\ \partial F & \hookrightarrow & T \end{array}$$